Computation for the lace expansion for percolation

Analysis of Section 3.5-3.6

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Abstract

In this file we performs the numerical part of the analysis of the non-backtracking lace expansion for percolation. All references in this version of the notebook will be to the PhD thesis of the author.

This file is accomanied by another notebook -SRW_Computations- where an number of simple random walks are computed. The user should first open that file, choose a dimension and execute all lines of the file. Then he is expects to choose constance Γ_i in this file. After choosing these quantities the used should select the menu item Evaluate-> Evaluate Notebook. In a table at the end of this document the result of the computations are shown. There it can be see whether the bootstrap with the given parameters and therefore the analysis was succesful. The computation of the -SRW_Computations- file are independent of the values Γ_i, so that the need to compute the SRW-integral once when we start the program and whenever we change the dimension.

We compute bound on the two-point function and repulsive diagrams (Section 5.1.2). We use these bounds to compute the Diagramatic bounds derived in Section 4.3. and compute the bounds used for the Analysis in Section 3.5.

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Input

We try to perform the bootstrap for the following values of Γ_i. So that the values of f_1, f_2,$\bar{f}_{3,n,l}$ and $\bar{f}_{4,n,l}$ ars small then:

In[155]:=
```
Gamma1 = 1.044379;
Gamma2 = 1.03712;
GammaFour[1, 4] = 0.00837843;

GammaThree[0, 0] = 0.14058;
GammaThree[1, 0] = 0.1979255;
GammaThree[1, 1] = 0.047;
GammaThree[1, 2] = 0.02301;
GammaThree[1, 3] = 0.00837828;
```

The bootstraps succeeds in dimension 15 with the constants (* deactivated *)

In[163]:=
```
(*Gamma1=1.044379;
Gamma2=1.03712;
GammaFour[1,4]=0.00837843;

GammaThree[0,0]=0.14058;
GammaThree[1,0]=0.1979255;
GammaThree[1,1]=0.047;
GammaThree[1,2]=0.02301;
GammaThree[1,3]=0.00837828;*)
```

Bound on the two-point function and on repulsive diagrams

Definition of Constants

We define the constants for two setting s: we use s=i for bound on $z = z_I$ and s=o for bound on zϵ $(z_I,\ z_c)$:

```
In[164]:= z[i] = 1/(2 d - 1);
z[o] = Gamma1/(2 d - 1);(* Upper bound on z and thereby also on z_c *)
(*bound on the two-point function*)
VarGamma1[i] = 1;
VarGamma1[o] = Gamma1;
VarGamma2[i] = (2 d - 2)/(2 d - 1); (*G_z(x) ≤B_z(x) ≤ (2d-2)/(2d-1) C(x)*)
VarGamma2[o] = Gamma2 * (2 d - 2)/(2 d - 1); (*Ĝ_z(k)≤ varGamma2 Ĉ(k), follows from f2*)
```

Further, we define variables to save the number of short NBWs, as given explain in Section 5.1.2 we can use the number if SAW only for SAW and LT:

```
In[170]:= c2ik = 2; (*c_2(e_1+e_2)*)
c4ik = 4 (2 d - 3) + 2 (2 d - 4);(*c_4(e_1+e_2)*)
c6ik = 16 + 84 (2 d - 4) + 36 (2 d - 4) (2 d - 6) + 6 d c3i;(*c_4(e_1+e_2)*)

c3i = (2 d - 2);(*c_3(e_1)*)
c5i = (3 (2 d - 2 ) + 4 (2 d - 2) (2 d - 4) ) + 4 d c3i;(*c_5(e_1)*)
c7i = (14 (2 d - 2 ) + 62 (2 d - 2) (2 d - 4) + 27 (2 d - 2) (2 d - 4) (2 d - 6) ) + 8 d c3i + 4 d c5i;
(*c_7(e_1)*)
```

Bounds on two-point function

We compute bounds as explained in Section 5.1.2. We begin by computing $G_{n,z}(e_1)$ as in (5.1.22)-(5.1.24)

```
In[176]:= Do[
 Bound[G7i, s] = c7i (z[s])^7 + (2 d z[s])^9 * VarGamma2[s] * I110; (* G_{7,z}(e_1)*)
 Bound[G5i, s] = c5i (z[s])^5 + Bound[G7i, s]; (* G_{5,z}(e_1)*)
 Bound[G3i, s] = c3i (z[s])^3 + Bound[G5i, s]; (* G_{3,z}(e_1)*)
 Bound[G1i, s] = z[s] + Bound[G3i, s]; (* G_{1,z}(e_1)*)
 rho[Lower, s] = 1 - Bound[G1i, s];
 rho[s] = 1 - z[i] - c3i z[i]^3 (1 - z[i]) - c5i z[i]^5 (1 - z[i])^2;
 , {s, {i, o}}]
```

Then we compute $G^1_{n,z}(e_1 + e_2)$ and $G^1_{4,z}(2\, e_2)$, see (5.1.25)-(5.1.26) :

```
In[177]:= Do[
 Bound[G8ik, s] = d/(d - 1) (2 d z[o])^8 VarGamma2[s] I110; (*G_8(e_1+e_2)*)
 Bound[G6ik, s] = c6ik z[s]^6 + VarGamma2[s] I18; (*G_6(e_1+e_2)*)
 Bound[G4ik, s] = Bound[G6ik, s] + (c4ik - 2 (2 d - 3) ) z[s]^4; (*G^1_4(e_1+e_2)*)
 Bound[G2ik, s] = Bound[G4ik, s] + (c2ik - 1) z[s]^2;(*G^1_2(e_1+e_2)*)

 , {s, {i, o}}]
```

We compute the supremum of the two-point function as given in (5.1.27)-(5.1.31):

In[178]:=
```
Do[
 Bound[G6, s] = Max[c6ik z[s]^6, c7i z[s]^7] + (2 d z[s])^8 VarGamma2[s] I18;
 (* Bound for sup_x G_6(x)=Max[G_7(e),G_6(e_1+e_2),sup_x G_8(x)]*)
 Bound[G4, s] = Max[c4ik z[s]^4, c5i z[s]^5] + Bound[G6, s]; (* Bound for sup_x G_4(x)*)
 Bound[G2, s] = Max[c2ik z[s]^2, c3i z[s]^3] + Bound[G4, s]; (* Bound for sup_x G_2(x)*)
 Bound[G1, s] = Max[Bound[G1i, s], Bound[G2, s]]; (* Bound for sup_x G_1(x)*)
 Bound[G4ii, s] = (2 d + 2) z[s]^4 + Bound[G6, s];(* Bound for sup_x G_4^1(2 e_1)*)
 , {s, {i, o}}]
```

Closed repulsive diagrams

We define the bounds on the closed repulsive diagrams as desribed in Section 5.1.2 in (5.1.34)-(5.1.36). The bound does only depent on the total number of steps and the number of tw-point functions involved. It does not depent on the indivitual length of the pieces m_1, m_2, ... and of the orientation of the arrows.

In[179]:=
```
Do[
 Bound[ClosedRepLoop, 4, s] = 2 d z[s] Bound[G3i, s];
 Bound[ClosedRepBubble, 4, s] =
  z[s]^4 (2 d c3i) + 3 z[s]^6 (2 d c5i) + 5 z[s]^8 (2 d c7i) + (2 d z[s])^10 VarGamma2[s] I110 +
   (2 d z[s])^10 VarGamma2[s]^2 I210;
 Bound[ClosedRepBubble, 3, s] =
  Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s];
 Bound[ClosedRepTriangle, 4, s] =
  ((4 + 1 - 4) (4 + 2 - 4))/2 z[s]^4 (2 d c3i) + ((6 + 1 - 4) (6 + 2 - 4))/2 z[s]^6 (2 d c5i) +
   ((8 + 1 - 4) (8 + 2 - 4))/2 z[s]^8 (2 d c7i) + ((10 - 6) (9 - 6))/2 (2 d z[s])^10 VarGamma2[s] I110 +
   6 (2 d z[s])^10 VarGamma2[s]^2 I210 + (2 d z[s])^10 VarGamma2[s]^3 I310;
 Bound[ClosedRepSquare, 4, s] =
  z[s]^4 (2 d c3i) + 10 z[s]^6 (2 d c5i) + 35 z[s]^8 (2 d c7i) +
   84 (2 d z[s])^10 VarGamma2[s] I110 + ((10 - 4) (9 - 4))/2 (2 d z[s])^10 VarGamma2[s]^2 I210 +
   6 (2 d z[s])^10 VarGamma2[s]^3 I310 + (2 d z[s])^10 VarGamma2[s]^4 I410;
 Bound[ClosedRepSquare, 3, s] =
  Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s] +
   Bound[ClosedRepTriangle, 4, s] + Bound[ClosedRepSquare, 4, s];
 Bound[ClosedRepSquare, 2, s] =
  Bound[ClosedRepLoop, 4, s] + 2 Bound[ClosedRepBubble, 3, s] +
   Bound[ClosedRepTriangle, 4, s] + Bound[ClosedRepSquare, 3, s];
 , {s, {i, o}}]
```

Open repulsive diagrams

Then, we define the bound on the open repulsive diagrams as in (5.3.38):

In[180]:=
```
Do[
 Bound[OpenRepBubble, 1, s] =
  Max[2 c2ik z[s]^2 + 4 c4ik z[s]^4 + 6 c6ik z[s]^6,
    z[s] + 3 c3i z[s]^3 + 5 c5i z[s]^5 + 7 c7i z[s]^7] + 7 (2 d z[s])^8 VarGamma2[s] I18 +
   (2 d z[s])^8 VarGamma2[s]^2 I28;
 Bound[OpenRepBubble, 2, s] =
  Max[c2ik z[s]^2 + 3 c4ik z[s]^4 + 5 c6ik z[s]^6, 2 c3i z[s]^3 + 4 c5i z[s]^5 + 6 c7i z[s]^7] +
   6 (2 d z[s])^8 VarGamma2[s] I18 + (2 d z[s])^8 VarGamma2[s]^2 I28;
 Bound[OpenRepBubble, 3, s] =
  Max[2 c4ik z[s]^4 + 4 c6ik z[s]^6, c3i z[s]^3 + 3 c5i z[s]^5 + 5 c7i z[s]^7] +
   5 (2 d z[s])^8 VarGamma2[s] I18 + (2 d z[s])^8 VarGamma2[s]^2 I28;
 Bound[OpenRepBubble, 4, s] =
  Max[c4ik z[s]^4 + 3 c6ik z[s]^6, 2 c5i z[s]^5 + 4 c7i z[s]^7] + 4 (2 d z[s])^8 VarGamma2[s] I18 +
   (2 d z[s])^8 VarGamma2[s]^2 I28;
 Bound[OpenRepTriangle, 1, s] =
  Max[3 c2ik z[s]^2 + 10 c4ik z[s]^4 + 21 c6ik z[s]^6,
    z[s] + 6 c3i z[s]^3 + 15 c5i z[s]^5 + 28 c7i z[s]^7] +
   (8 - 1) (7 - 1)/2 (2 d z[s])^8 VarGamma2[s] I18 + 7 (2 d z[s])^8 VarGamma2[s]^2 I28 +
   (2 d z[s])^8 VarGamma2[s]^3 I38;
 Bound[OpenRepTriangle, 2, s] =
  Max[c2ik z[s]^2 + 6 c4ik z[s]^4 + 15 c6ik z[s]^6, 3 c3i z[s]^3 + 10 c5i z[s]^5 + 21 c7i z[s]^7] +
   (8 - 2) (7 - 2)/2 (2 d z[s])^8 VarGamma2[s] I18 + 6 (2 d z[s])^8 VarGamma2[s]^2 I28 +
   (2 d z[s])^8 VarGamma2[s]^3 I38;
 Bound[OpenRepTriangle, 3, s] =
  Max[3 c4ik z[s]^4 + 10 c6ik z[s]^6, c3i z[s]^3 + 6 c5i z[s]^5 + 15 c7i z[s]^7] +
   (8 - 3) (7 - 3)/2 (2 d z[s])^8 VarGamma2[s] I18 + 5 (2 d z[s])^8 VarGamma2[s]^2 I28 +
   (2 d z[s])^8 VarGamma2[s]^3 I38;
 Bound[OpenRepTriangle, 4, s] =
  Max[c4ik z[s]^4 + 6 c6ik z[s]^6, 3 c5i z[s]^5 + 10 c7i z[s]^7] +
   (8 - 4) (7 - 4)/2 (2 d z[s])^8 VarGamma2[s] I18 + 5 (2 d z[s])^8 VarGamma2[s]^2 I28 +
   (2 d z[s])^8 VarGamma2[s]^3 I38;
 Bound[OpenRepSquare, 2, s] =
  Max[c2ik z[s]^2 + 10 c4ik z[s]^4 + 35 c6ik z[s]^6, 3 c3i z[s]^3 + 20 c5i z[s]^5 + 56 c7i z[s]^7] +
   84 (2 d z[s])^8 VarGamma2[s] I18 + (8 - 2) (7 - 2)/2 (2 d z[s])^8 VarGamma2[s]^2 I28 +
   6 (2 d z[s])^8 VarGamma2[s]^3 I38 + (2 d z[s])^8 VarGamma2[s]^4 I48;
 , {s, {i, o}}]
```

Weighted Diagrams

First we define the bound on the weighted diagams in the same format as we used the in the implementation for the analysis of Section 3.5.

For $z = z_i$ we use the bound (3.6.20) :

In[181]:=
```
Bound[WeightedOpenLine, 0, i] = BoundFThreeBarInitial[0, 0, rho[i]];
Bound[WeightedClosedBubble, 4, i] =
  (2 d z [i])^4 BoundFFourBarInitial[1, 4, rho[i]];
Do[
 Bound[WeightedOpenBubble, t, i] = (2 d z [i])^t BoundFThreeBarInitial[1, t, rho[i]];
 , {t, 0, 3}]
```

Then we use $f_{3,n,l}$ and $f_{4,n,l}$ that gives us direct bound for the weighted diagrams for $z \in (z_i, z_c)$:

In[184]:=
```
Bound[WeightedOpenLine, 0, o] = GammaThree[0, 0];
Bound[WeightedClosedBubble, 4, o] = (2 d z [i])^4 GammaFour[1, 4];

Do[
 Bound[WeightedOpenBubble, t, o] = (2 d z [i])^t GammaThree[1, t];
 , {t, 0, 3}]
```

Then we use the idea explained in (5.1.42)-(5.1.49) to obtain

In[187]:=
```
Do[
 Bound[WeightedClosedBubble, 3, s] =
  Bound[WeightedClosedBubble, 4, s] +
   2 d z[s]^3 (Bound[G3i, s] + 3 (2 d - 2) d/(d - 1) d/(d - 2) (2 d z[o])^3 VarGamma2[s] I16 +
      5 (2 d - 2) d/(d - 1) Bound[G4, s] + 9 (z[s]^3 + 3 (2 d - 2) z[s]^5 + Bound[G6, s]));

 Bound[WeightedClosedBubble, 2, s] =
  Bound[WeightedClosedBubble, 3, s] + 8 d z[s]^2 Bound[G4ii, s] +
   8 d z[s]^2 (2 d - 2) (z[s]^2 + Bound[G4ik, s]);
 Bound[WeightedClosedBubble, 0, s] =
  Bound[WeightedClosedBubble, 2, s] + 2 d z[s] Bound[G3i, s];
 , {s, {i, o}}]
```

Definition of diagrams without weight

Here we define the quantities of Section 4.5.2. The element of the bounds (4.5.30)-(4.5.38). We defined the element in the thesis using bubble, triangles, square and even pentragram. The bound on these diagrams depend only on the number of two-point funtions/pieces without fix lengh and the number of fixed steps.

We define the bound on $P^{(0),b}$define in Table 4.18 as follows:

In[188]:=
```
Do[
  Bound[P, 0, s] = 1 + Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s];
  Bound[P, 1, s] = 2 Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s];
  Bound[P, 2, s] = Bound[ClosedRepLoop, 4, s] +
    Bound[ClosedRepBubble, 4, s] + Bound[ClosedRepTriangle, 4, s];
  , {s, {i, o}}];
```

The bound on $A^{a,b}$define in Table 4.19 we declare the variables:

In[189]:=
```
Do[
  Bound[A, 0, 0, s] = Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s];
  Bound[A, 0, 1, s] = Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s];
  Bound[A, 0, 2, s] = Bound[ClosedRepTriangle, 4, s];
  Bound[A, 1, 0, s] =
   1/(2 d z[s]) (Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s]);
  Bound[A, 1, 1, s] =
   1/(2 d z[s]) (Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s]);
  Bound[A, 1, 2, s] = Bound[ClosedRepTriangle, 4, s]/(2 d z[s]);
  Bound[A, 2, 0, s] = Bound[OpenRepBubble, 1, s];
  Bound[A, 2, 1, s] = Bound[OpenRepBubble, 2, s];
  Bound[A, 2, 2, s] = Bound[OpenRepTriangle, 3, s];
  , {s, {i, o}}];
```

We defined $A^{i,a,b}$ using diagrams what do not need to be repulsive. Follow Table 4.20 we define

In[190]:=
```
Do[
  Bound[Ai, 0, 0, s] = Bound[ClosedRepBubble, 3, s];
  Bound[Ai, 0, 1, s] = (2 d z[s])^4 VarGamma2[s] I14 + (2 d z[s])^4 VarGamma2[s]^2 I24;
  Bound[Ai, 0, 2, s] = (2 d z[s])^4 VarGamma2[s]^3 I34;
  Bound[Ai, 1, 0, s] = ((2 d z[s])^4 VarGamma2[s] I14 + (2 d z[s])^4 VarGamma2[s]^2 I24)/(2 d z[s]);
  Bound[Ai, 1, 1, s] = ((2 d z[s])^4 VarGamma2[s] I14 + (2 d z[s])^4 VarGamma2[s]^2 I24)/(2 d z[s]);
  Bound[Ai, 1, 2, s] = (2 d z[s])^3 VarGamma2[s]^3 I34;
  Bound[Ai, 2, 0, s] = (2 d z[s])^2 VarGamma2[s]^2 I22;
  Bound[Ai, 2, 1, s] = (2 d z[s])^2 VarGamma2[s]^2 I22;
  Bound[Ai, 2, 2, s] = (2 d z[s])^3 VarGamma2[s]^3 Sqrt[I32 I34];
  , {s, {i, o}}];
```

Then we define $\overline{A}^{a,b}$ as in (4.5.16)-(4.5.20):

In[191]:=
```
Do[
  Do[
   Bound[Abar, a, 0, s] = Bound[Ai, a, 0, s];
   Bound[Abar, a, 1, s] = Bound[Ai, a, 1, s]/(2 d z[i]);
   , {a, {0, 1, 2}}];
  Bound[Abar, 0, 2, s] = (2 d z[s])^2 VarGamma2[s]^2 I22;
  Bound[Abar, 1, 2, s] = (2 d z[s]) VarGamma2[s]^2 I22;
  Bound[Abar, 2, 2, s] = VarGamma2[s]^2 I20;
  , {s, {i, o}}];
```

Then we define $B^{2,i,a,b}$ as in Table 4.21 and the comment after (4.5.20)

```
In[192]:= Do[
  Do[
   Bound[B2i, a, 0, s] = 0;
   Bound[B2i, a, 1, s] = 0;
   , {a, {0, 1, 2}}];
  Bound[B2i, 0, 2, s] = Bound[ClosedRepBubble, 3, s] Bound[ClosedRepTriangle, 4, s] / (2 d z[s]) +
    Bound[ClosedRepTriangle, 4, s] Bound[OpenRepTriangle, 3, s];
  Bound[B2i, 1, 2, s] = Bound[ClosedRepBubble, 3, s] / (2 d z[s]) Bound[ClosedRepTriangle, 4, s] / (2 d z[s]) +
    Bound[ClosedRepTriangle, 4, s] Bound[OpenRepTriangle, 3, s] / (2 d z[s]);
  Bound[B2i, 2, 2, s] = Bound[ClosedRepBubble, 3, s] / (2 d z[s]) Bound[OpenRepTriangle, 3, s] +
    Bound[OpenRepBubble, 2, s] Bound[OpenRepTriangle, 4, s];
  , {s, {i, o}}];
```

Further, we define $\overline{B}^{2,i,a,b}$ as in Tables 4.22-4.23 and the comment after (4.5.20)

```
In[193]:= Do[
  Do[
   Bound[Bbar2i, 0, a, s] = 0;
   , {a, {0, 1, 2}}];
  Bound[Bbar2i, 1, 0, s] = Bound[G3i, s] Bound[ClosedRepBubble, 4, s] +
    Bound[G2, s] Bound[ClosedRepTriangle, 4, s];
  Bound[Bbar2i, 1, 1, s] = Bound[G2, s] Bound[ClosedRepTriangle, 4, s] / (2 d z[s]);
  Bound[Bbar2i, 1, 2, s] = Bound[G3i, s] Bound[OpenRepBubble, 3, s] +
    Bound[G2, s] Bound[OpenRepTriangle, 4, s];
  Bound[Bbar2i, 2, 0, s] = Bound[ClosedRepBubble, 3, s] Bound[OpenRepBubble, 4, s] +
    Bound[ClosedRepBubble, 3, s] Bound[ClosedRepTriangle, 4, s] +
    Bound[ClosedRepTriangle, 4, s] Bound[OpenRepTriangle, 3, s];
  Bound[Bbar2i, 2, 1, s] = Bound[Bbar2i, 2, 0, s] / (2 d z[s]);
  Bound[Bbar2i, 2, 2, s] =
   Bound[G1, o] Bound[OpenRepBubble, 2, s] Bound[ClosedRepBubble, 3, s] +
    Bound[OpenRepTriangle, 3, s] Bound[ClosedRepBubble, 3, s] / (2 d z[s]) +
    Bound[G1, o] Bound[OpenRepBubble, 1, s] Bound[ClosedRepTriangle, 4, s];
  , {s, {i, o}}];
```

We define the bound on $P^{(N),i,b}$define in Table 4.24 as follows:

```
In[194]:= Do[
    Bound[Piota, 0, s] = 2 d Bound[G3i, s] Bound[P, 0, s];
    Bound[Piota, 1, s] = 2 d Bound[G3i, s] Bound[P, 1, s] + 2 d (z[s] + Bound[G3i, s]);
    Bound[Piota, 2, s] = 2 d Bound[G3i, s] Bound[P, 2, s] + Bound[G3i, s] +
      Bound[ClosedRepBubble, 3, s] / z[s] +
      2 d (z[s] + Bound[G3i, s] + Bound[ClosedRepBubble, 3, s] / (2 d z[s]))
       Bound[ClosedRepBubble, 3, s];
    , {s, {i, o}}];
```

Definition of diagrams with weight

Now we first implement the bound in the diagrams $H^{1,a,b}$, $H^{2,i,a,b}$ and $H^{3,i,a,b}$ defined in (4.5.23)-(4.5.25):

In[195]:=
```
Do[
  Bound[H1, 0, 0, s] = Bound[WeightedClosedBubble, 0, s];
  Bound[H1, 1, 0, s] = Bound[WeightedClosedBubble, 0, s] / (2 d z[s]);
  Bound[H1, 0, 1, s] = Bound[WeightedClosedBubble, 0, s] / (2 d z[s]);
  Bound[H1, 1, 1, s] = Bound[WeightedClosedBubble, 2, s] / (2 d z[s])^2;
  Bound[H1, 2, 0, s] = Bound[WeightedOpenBubble, 0, s];
  Bound[H1, 0, 2, s] = Bound[WeightedOpenBubble, 0, s];

  Bound[H1, 2, 1, s] = Bound[WeightedOpenBubble, 1, s] / (2 d z[s]);
  Bound[H1, 1, 2, s] = Bound[WeightedOpenBubble, 1, s] / (2 d z[s]);
  Bound[H1, 2, 2, s] = Bound[WeightedOpenBubble, 0, s];

  Do[Do[
    Bound[H2, a, b, s] = Bound[H1, a, b, s];
    , {a, {0, 1, 2}}], {b, {0, 1, 2}}];
  (* for H3 we know that the unweighted path has at least length one*)
  Bound[H3, 0, 0, s] = Bound[WeightedClosedBubble, 0, s];
  Bound[H3, 1, 0, s] = Bound[WeightedClosedBubble, 2, s] / (2 d z[s]);
  Bound[H3, 0, 1, s] = Bound[WeightedClosedBubble, 2, s] / (2 d z[s]);
  Bound[H3, 1, 1, s] = Bound[WeightedClosedBubble, 3, s] / (2 d z[s])^2;
  (* to use the information that the unweighted path has at least length
   one we use Chauchy schwarz to obtain*)
  Bound[H3, 2, 0, s] = Bound[WeightedOpenBubble, 1, s];
  Bound[H3, 0, 2, s] = Bound[WeightedOpenBubble, 1, s];

  Bound[H3, 1, 2, s] = Bound[WeightedOpenBubble, 2, s] / (2 d z[o]);
  Bound[H3, 2, 1, s] = Bound[WeightedOpenBubble, 2, s] / (2 d z[o]);
  Bound[H3, 2, 2, s] = Bound[WeightedOpenBubble, 1, s];

  Clear[a, b, t];
  , {s, {i, o}}]
```

As explained in Section 4.5.6 we bound $C^{1,i,a,b}$ and $C^{2,i,a,b}$ in terms of other diagram. At this point we only implement the bound on $C^{3,i,a,b}$, see (4.5.90)-(4.5.94). For a=2,b=2 we implement the bound as describes in (4.5.97). For other a,b we improve this bound using three informations: 1.) By symmetrie can we bound the contribution $C^{3,i,a,b}$ in the same way as $C^{3,i,b,a}$. 2.) if a or b are 1 then we can use symmetrie to create an extra $\hat{D}(k)$. 3.) the complete square consists of at least four steps to improve the bounds. Using these three points to obtain the bounds:

In[196]:=
```
Do[

  Bound[C3, 2, 2, s] = Bound[WeightedOpenLine, 0, s] (2 d z[s])^2 VarGamma2[s]^3
    I32 Bound[OpenRepSquare, 2, s];
  Bound[C3, 1, 2, s] = Bound[WeightedOpenLine, 0, s] (2 d z[s])^2 VarGamma2[s]^3
    I32 Bound[ClosedRepSquare, 3, s] / (2 d z[s]);
  Bound[C3, 2, 1, s] = Bound[C3, 1, 2, s];
  Bound[C3, 1, 1, s] = Bound[WeightedOpenLine, 0, s] Bound[OpenRepTriangle, 3, s]
    Bound[ClosedRepSquare, 3, s] / (2 d z[s]);
  Bound[C3, 0, 1, s] = Bound[WeightedOpenLine, 0, s] Bound[OpenRepTriangle, 2, s]
    Bound[ClosedRepSquare, 3, s] / (2 d z[s]);
  Bound[C3, 1, 0, s] = Bound[C3, 0, 1, s];
  Bound[C3, 0, 0, s] = Bound[WeightedOpenLine, 0, s] Bound[OpenRepTriangle, 2, s]
    Bound[ClosedRepSquare, 2, s];
  Bound[C3, 0, 2, s] = Bound[WeightedOpenLine, 0, s] (2 d z[s])^2 VarGamma2[s]^3
    I32 Bound[ClosedRepSquare, 2, s];
  Bound[C3, 2, 0, s] = Bound[C3, 0, 2, s];
  , {s, {i, o}}]
```

Definition of diagrams specific for initial step iota

As next we bound the term defined in (4.5.28) and bound in (4.5.100)-(4.5.102). We improve the bound stated of the first diagram of Figure 4.29, by noting that y=0 is not possible, so that the unweighted lines has always at least 2 steps.

In[197]:=
```
Do[

  Bound[hi, part1, 0, s] = Bound[WeightedClosedBubble, 2, s];
  Bound[hi, part1, 1, s] = Bound[WeightedClosedBubble, 2, s] / (2 d z[s]);
  Bound[hi, part1, 2, s] = Bound[WeightedOpenBubble, 1, s];
  Do[
   Bound[hi, part2, b, s] = 2 d Bound[G3i, s] Bound[H2, 0, b, s] +
     4 d Bound[G3i, s]
      (Sum[Bound[P, a, s] Bound[H2, a, b, s], {a, 0, 2}] - Bound[H2, 0, b, s]) +
     4 d Bound[G3i, s] Sum[ Bound[H1, 0, a, s] Bound[Ai, a, b, s], {a, 0, 2}];
   Bound[hi, part3, b, s] = Bound[ClosedRepLoop, 4, s] / (2 d z[s]) Bound[H2, 1, b, s] +
     Bound[ClosedRepBubble, 3, s] / (2 d z[s]) Bound[H2, 2, b, s] +
     2 (Bound[ClosedRepLoop, 4, s] / (2 d z[s]) + Bound[ClosedRepBubble, 3, s] / (2 d z[s]))
      (Bound[Ai, 0, 0, s] Max[Bound[H2, 1, b, s], Bound[H2, 2, b, s]] +
        Bound[H1, 0, 0, s] Max[Bound[Ai, 1, b, s], Bound[Ai, 2, b, s]]);

   Bound[hi, b, s] = Sum[Bound[hi, t, b, s], {t, {part1, part2, part3}}]
   , {b, 0, 2}];
  , {s, {i, o}}]
```

Then we define the bound for $h^{i,\mathrm{II},b}$ defined in (4.5.29) and bounded in (4.5.99).

In[198]:=
```
Do[
 Do[
   Bound[hII, b, s] =
    Bound[hi, b, s] +
     2 Sum[Bound[hi, a, s] Bound[Ai, a, b, s] +
        Sum[Bound[Piota, a, s] Bound[Ai, a, c, s] Bound[H2, c, b, s], {c, 0, 2}],
       {a, 0, 2}]
   , {b, 0, 2}];
 , {s, {i, o}}]
```

Definition of vectors and matrices

Then we define the matricies we use the compute/state the bounds. First we define the matricies for which we have already computed the entries.

In[199]:=
```
Do[
 Vector[P, s] = Table[Bound[P, a, s], {a, {0, 1, 2}}];
 Vector[PNT, s] = Vector[P, s] - {1, 0, 0};
 Vector[Piota, s] = Table[Bound[Piota, a, s], {a, {0, 1, 2}}];

 Vector[h, s] = Table[Bound[H1, 0, b, s], {b, {0, 1, 2}}];
 Vector[hi, s] = Table[Bound[hi, b, s], {b, {0, 1, 2}}];
 Vector[hII, s] = Table[Bound[hII, b, s], {b, {0, 1, 2}}];

 Matrix[A, s] = Table[Bound[A, a, b, s], {a, {0, 1, 2}}, {b, {0, 1, 2}}];
 Matrix[Ai, s] = Table[Bound[Ai, a, b, s], {a, {0, 1, 2}}, {b, {0, 1, 2}}];
 Matrix[Abar, s] = Table[Bound[Abar, a, b, s], {a, {0, 1, 2}}, {b, {0, 1, 2}}];
 Matrix[B2i, s] = Table[Bound[B2i, a, b, s], {a, {0, 1, 2}}, {b, {0, 1, 2}}];
 Matrix[Bbar2i, s] = Table[Bound[Bbar2i, a, b, s], {a, {0, 1, 2}}, {b, {0, 1, 2}}];

 Matrix[C3, s] = Table[Bound[C3, a, b, s], {a, {0, 1, 2}}, {b, {0, 1, 2}}];
 Matrix[H1, s] = Table[Bound[H1, a, b, s], {a, {0, 1, 2}}, {b, {0, 1, 2}}];
 Matrix[H2, s] = Table[Bound[H2, a, b, s], {a, {0, 1, 2}}, {b, {0, 1, 2}}];
 Matrix[H3, s] = Table[Bound[H3, a, b, s], {a, {0, 1, 2}}, {b, {0, 1, 2}}];
 , {s, {i, o}}]
```

Then we define the bound for B and $\overline{B}$, see (4.5.21)-(4.5.22)

In[200]:=
```
Do[
 Matrix[B, s] = Matrix[Ai, s].Matrix[A, s] + Matrix[Ai, s] * Bound[P, 0, s] +
   Matrix[B2i, s];
 Matrix[Bbar, s] = Matrix[Ai, s].Matrix[A, s] + Matrix[Ai, s] + Matrix[Bbar2i, s];
 , {s, {i, o}}]
Clear[a, b]
```

Further, we define the bound on C^1 and C^2 as stated in (4.5.90) and (4.5.92)

In[202]:=
```
Do[
 Matrix[C1, s] =
  (2 Matrix[H2, s].Matrix[A, s] + 2 Matrix[Ai, s].Matrix[H1, s] + Matrix[H2, s]).
    Matrix[Ai, s] + Matrix[H2, s].Matrix[Bbar2i, s];
 Matrix[C2, s] =
  (2 Matrix[H3, s].Matrix[A, s] + 2 Matrix[Ai, s].Matrix[H1, s] + Matrix[H3, s]).
    Matrix[Ai, s] + 2 Matrix[C3, s] + Matrix[H3, s].Matrix[Bbar2i, s];
 , {s, {i, o}}]
Clear[a, b]
```

We compute the eigensystem of the maticies B and $\overline{B}$ to sum the bounds as explained in Section 5.3.

```
In[204]:= Do[
  EigensystemB[s] = Eigensystem[Transpose[Matrix[B, s]]];
  EigensystemBbar[s] = Eigensystem[Matrix[Bbar, s]];
  InverseProductB[s] = Inverse[Transpose[EigensystemB[s][[2]]]].Vector[P, s];
  InverseProductBForPIota[s] =
   Inverse[Transpose[EigensystemB[s][[2]]]].Vector[Piota, s];
  InverseProductBbar[s] = Inverse[Transpose[EigensystemBbar[s][[2]]]].Vector[P, s];

  Do[
   EigenVectorB[j, s] = EigensystemB[s][[2, j]] * InverseProductB[s][[j]];
   EigenVectorBbar[j, s] = EigensystemBbar[s][[2, j]] * InverseProductBbar[s][[j]];
   EigenVectorBForPIota[j, s] =
    EigensystemB[s][[2, j]] * InverseProductBForPIota[s][[j]];

   EigenValueB[j, s] = EigensystemB[s][[1, j]];
   EigenValueBbar[j, s] = EigensystemBbar[s][[1, j]];
   EigenValueBForPIota[j, s] = EigensystemB[s][[1, j]];
   , {j, 1, 3}];
  , {s, {i, o}}]
 Clear[a, b]
```

Bound on the coefficients

Bound for k=0

Then we implement the bounds stated in Lemma 4.5.1.-4.5.2 and Proposition 4.5.3. explixitely treat N=2,3 as they will also recieve special attentention when we compute the bound on Ξ(0)-Ξ(k)

```
In[206]:= Do[
  Bound[Xi, normal, 0, s] = Bound[P, 0, s] - 1;
  (* For the analysis we explicitly extract the contribution of δ_{0,x}*)
  Bound[Xi, iota, 0, s] = Bound[G3i, s] Bound[P, 0, s];

  Bound[Xi, normal, 1, s] = Vector[P, s].Matrix[Abar, s].Vector[P, s];
  Bound[Xi, iota, 1, s] = 1/(2 d) Vector[Piota, s].Matrix[Abar, s].Vector[P, s];

  Bound[Xi, normal, 2, s] = Vector[P, s].Matrix[B, s].Matrix[Abar, s].Vector[P, s];
  Bound[Xi, iota, 2, s] =
   1/(2 d) Vector[Piota, s].Matrix[B, s].Matrix[Abar, s].Vector[P, s];

  Bound[Xi, normal, 3, s] = Vector[P, s].Matrix[B, s].Matrix[B, s].
    Matrix[Abar, s].Vector[P, s];
  Bound[Xi, iota, 3, s] =
   1/(2 d) Vector[Piota, s].Matrix[B, s].Matrix[B, s].Matrix[Abar, s].Vector[P, s];

  , {s, {i, o}}]
```

We compue the sum over all odd and even N=> 4

In[207]:=
```
Do[
 Bound[Xi, normal, EvenTail, s] =
  Sum[EigenValueB[j, s]^3/(1 - EigenValueB[j, s]^2) EigenVectorB[j, s].Matrix[Abar, s].Vector[P, s],
   {j, {1, 2, 3}}];
 Bound[Xi, normal, OddTail, s] =
  Sum[EigenValueB[j, s]^4/(1 - EigenValueB[j, s]^2) EigenVectorB[j, s].Matrix[Abar, s].Vector[P, s],
   {j, {1, 2, 3}}];
 Bound[Xi, iota, EvenTail, s] =
  1/(2 d)
   Sum[EigenValueBForPIota[j, s]^3/(1 - EigenValueBForPIota[j, s]^2)
     EigenVectorBForPIota[j, s].Matrix[Abar, s].Vector[P, s], {j, {1, 2, 3}}];
 Bound[Xi, iota,  OddTail, s] =
  1/(2 d) Sum[EigenValueBForPIota[j, s]^4/(1 -  EigenValueBForPIota[j, s]^2)
     EigenVectorBForPIota[j, s].Matrix[Abar, s].Vector[P, s], {j, {1, 2, 3}}];
 , {s, {i, o}}]
```

Then we compute the sum over all odd/even N

In[208]:=
```
Do[
 Do[
  Bound[Xi, a, Even, s] = Sum[Bound[Xi, a, t, s], {t, {0, 2, EvenTail}}];
  Bound[Xi, a, Odd, s] = Sum[Bound[Xi, a, t, s], {t, {1, 3, OddTail}}];
  Bound[Xi, a, Absolut, s] = Sum[Bound[Xi, a, t, s], {t, {Odd, Even}}];
  (*Print[Bound[Xi,a, Absolut, s]];*)
  , {a, {normal, iota}}]
 , {s, {i, o}}]
```

Bound for 0 - k

Bounds for $\hat{\Xi}(0)$-$\hat{\Xi}(k)$

First we compute the terms for N=0,1,2,3 and extract therefore the contributions with trivial first and last triangle

In[209]:=

```
Do[
 Bound[Xi, normal, 0, Delta, 0, s] = Bound[WeightedClosedBubble, 0, s];
 Bound[Xi, normal, 1, Delta, 0, s] =
  Bound[H2, 0, 0, s] + 4 Sum[Bound[H2, 0, b, s] Bound[Ai, 0, b, s], {b, 0, 2}] +
   6 Vector[h, s].Matrix[Ai, s].Vector[PNT, s] +
   3 Vector[PNT, s].Matrix[H2, s].Vector[PNT, s];
 (*for this bound we use the spatial symmetrie of the diagrams,
 when one triangle it trivial to reduce the factor 8 stated in (4.5.47)
  to 4 (sinis terms cancel, so we can split the cosin without creating
    the factor 2)*)
 Bound[Xi, normal, 2, Delta, 0, s] =
  4 (Vector[h, s].Matrix[Bbar, s].Matrix[Ai, s].Vector[PNT, s] +
      Vector[PNT, s].Matrix[B, s].
       (Matrix[H3, s].Vector[PNT, s] + Matrix[Ai, s].Vector[h, s]) +
      Vector[PNT, s].Matrix[C1, s].Vector[PNT, s]) +
   6 (Vector[h, s].Matrix[Bbar, s].Matrix[Ai, s].{1, 0, 0} +
      Vector[PNT, s].Matrix[B, s].Matrix[H2, s].{1, 0, 0} +
      Vector[PNT, s].Matrix[C1, s].{1, 0, 0}) +
   2 ({1, 0, 0}.Matrix[B, s].Matrix[H2, s].{1, 0, 0} +
      {1, 0, 0}.Matrix[C1, s].{1, 0, 0});
 Bound[Xi, normal, 3, Delta, 0, s] =
  5 (Vector[h, s].Matrix[Bbar, s].Matrix[Bbar, s].Matrix[Ai, s].Vector[PNT, s] +
      Vector[PNT, s].Matrix[B, s].Matrix[B, s].
       (Matrix[H2, s].Vector[PNT, s] + Matrix[Ai, s].Vector[h, s]) +
      Vector[PNT, s].(Matrix[C1, s].Matrix[Bbar, s] + Matrix[B, s].Matrix[C2, s]).
       Vector[PNT, s]) +
   4
    ({1, 0, 0}.Matrix[B, s].Matrix[B, s].
       (Matrix[H2, s].Vector[PNT, s] + Matrix[Ai, s].Vector[h, s]) +
      {1, 0, 0}.(Matrix[C1, s].Matrix[Bbar, s] + Matrix[B, s].Matrix[C2, s]).
       Vector[PNT, s]) +
   4 (Vector[h, s].Matrix[Bbar, s].Matrix[Bbar, s].Matrix[Ai, s].{1, 0, 0} +
      Vector[PNT, s].Matrix[B, s].Matrix[B, s].(Matrix[H2, s].{1, 0, 0}) +
      Vector[PNT, s].(Matrix[C1, s].Matrix[Bbar, s] + Matrix[B, s].Matrix[C2, s]).
       {1, 0, 0}) +
   3 ({1, 0, 0}.Matrix[B, s].Matrix[B, s].Matrix[H2, s].{1, 0, 0} +
      {1, 0, 0}.(Matrix[C1, s].Matrix[Bbar, s] + Matrix[B, s].Matrix[C2, s]).
       {1, 0, 0});
 , {s, {i, o}}]
```

Then we compute the sum over the remaining N

In[210]:=

```
Do[
 Do[
  v[j] = EigenVectorB[j, s];
  vb[j] = EigenVectorBbar[j, s];
  e[j] = EigenValueB[j, s];
  eb[j] = EigenValueBbar[j, s];
  , {j, {1, 2, 3}}];
 Bound[Xi, normal, EvenTail, Delta, 0, s] =
  Vector[h, s].Matrix[Ai, s].Sum[vb[j] * eb[j]^3 (2/(1 - eb[j]^2)^2 + 4/(1 - eb[j]^2)),
```

```
    {j, {1, 2, 3}}] + Sum[e[j]^3 (2/(1 - e[j]^2)^2 + 4/(1 - e[j]^2)) * v[j], {j, {1, 2, 3}}].
   (Matrix[H3, s].Vector[P, s] + Matrix[Ai, s].Vector[h, s]) +
  2 Sum[v[j] 1/(1 - e[j]^2), {j, {1, 2, 3}}].Matrix[C1, s].
   Sum[vb[j] 1/(1 - eb[j]^2)^2, {j, {1, 2, 3}}] +
  2 Sum[v[j] 1/(1 - e[j]^2)^2, {j, {1, 2, 3}}].Matrix[C1, s].
   Sum[vb[j] 1/(1 - eb[j]^2), {j, {1, 2, 3}}] - 4 Vector[P, s].Matrix[C1, s].Vector[P, s] +
  2 Sum[v[j] e[j]/(1 - e[j]^2), {j, {1, 2, 3}}].Matrix[C2, s].
   Sum[vb[j] eb[j] (1/(1 - eb[j]^2)^2 + 1/(1 - eb[j]^2)), {j, {1, 2, 3}}] +
  2 Sum[v[j] e[j]/(1 - e[j]^2)^2, {j, {1, 2, 3}}].Matrix[C1, s].
   Sum[vb[j] eb[j] 1/(1 - eb[j]^2), {j, {1, 2, 3}}];

Bound[Xi, normal, OddTail, Delta, 0, s] =
 Vector[h, s].Matrix[Ai, s].Sum[vb[j] * eb[j]^4 (2/(1 - eb[j]^2)^2 + 5/(1 - eb[j]^2)),
    {j, {1, 2, 3}}] + Sum[e[j]^4 (2/(1 - e[j]^2)^2 + 5/(1 - e[j]^2)) * v[j], {j, {1, 2, 3}}].
   (Matrix[H3, s].Vector[P, s] + Matrix[Ai, s].Vector[h, s]) +
  Sum[v[j] 2/(1 - e[j]^2)^2, {j, {1, 2, 3}}].
   (Matrix[C2, s].Matrix[Bbar, s] + Matrix[B, s].Matrix[C1, s]).
   Sum[vb[j] 1/(1 - eb[j]^2), {j, {1, 2, 3}}] +
  Sum[v[j] 1/(1 - e[j]^2), {j, {1, 2, 3}}].
   (Matrix[C2, s].Matrix[Bbar, s] + Matrix[B, s].Matrix[C1, s]).
   Sum[vb[j] (2/(1 - eb[j]^2)^2 + 1/(1 - eb[j]^2)), {j, {1, 2, 3}}] +
  5 Vector[P, s].(Matrix[C2, s].Matrix[Bbar, s] + Matrix[B, s].Matrix[C1, s]).
    Vector[P, s];

, {s, {i, o}}]
```

Bounds for $\sum_i \hat{\Xi}^i(0)-\hat{\Xi}^i(k)$

First we compute the terms for N=0,1,2,3 and therefore extract the contributions with trivial first and last triangle

In[211]:=
```
Do[
 Bound[Xi, iota, 0, Delta, ei, s] =
  2 d (2 d - 1) z[s] Bound[G3i, s]^2 +
   (2 d - 1) Bound[G3i, s] Bound[WeightedClosedBubble, 2, s];
 Bound[Xi, iota, 0, Delta, 0, s] =
  Bound[Xi, iota, 0, Delta, ei, s] + 2 d Bound[Xi, iota, 0, s];
 (* we can use symmetrie to cancel the sin terms an obtain this bound*)

 Bound[Xi, iota, 1, Delta, ei, s] =
  Bound[hi, 0, s] + Vector[hi, s].Vector[PNT, s] +
   Vector[Piota, s].Matrix[Ai, s].Vector[h, s];
 Bound[Xi, iota, 1, Delta, 0, s] =
  Bound[Xi, iota, 1, Delta, ei, s] + 2 d Bound[Xi, iota, 1, s];

 Bound[Xi, iota, 2, Delta, ei, s] =
  2 (Vector[hII, s].Matrix[Bbar, s].{1, 0, 0} +
      Vector[Piota, s].Matrix[B, s].Matrix[H3, s].{1, 0, 0}) +
   3 (Vector[hII, s].Matrix[Bbar, s].Vector[PNT, s] +
      Vector[Piota, s].Matrix[B, s].
       (Matrix[H3, s].Vector[PNT, s] + Matrix[Ai, s].Vector[h, s]));
 Bound[Xi, iota, 2, Delta, 0, s] =
  3 (Vector[hII, s].Matrix[Bbar, s].{1, 0, 0} +
      Vector[Piota, s].Matrix[B, s].Matrix[H3, s].{1, 0, 0}) +
   4 (Vector[hII, s].Matrix[Bbar, s].Vector[PNT, s] +
      Vector[Piota, s].Matrix[B, s].
       (Matrix[H3, s].Vector[PNT, s] + Matrix[Ai, s].Vector[h, s])) +
   3 Vector[Piota, s].Matrix[B, s].Matrix[Abar, s].{1, 0, 0} +
   4 Vector[Piota, s].Matrix[B, s].Matrix[Abar, s].Vector[PNT, s] ;

 Bound[Xi, iota, 3, Delta, ei, s] =
  3 (Vector[hII, s].Matrix[Bbar, s].Matrix[Bbar, s].{1, 0, 0} +
      Vector[Piota, s].Matrix[B, s].Matrix[B, s].Matrix[H3, s].{1, 0, 0}) +
   3 Vector[Piota, s].Matrix[B, s].Matrix[C2, s].{1, 0, 0} +
   4 (Vector[hII, s].Matrix[Bbar, s].Matrix[Bbar, s].Vector[PNT, s] +
      Vector[Piota, s].Matrix[B, s].Matrix[B, s].
       (Matrix[H3, s].Vector[PNT, s] + Matrix[Ai, s].Vector[h, s])) +
   4 Vector[Piota, s].Matrix[B, s].Matrix[C2, s].Vector[PNT, s];
 Bound[Xi, iota, 3, Delta, 0, s] =
  4 (Vector[hII, s].Matrix[Bbar, s].Matrix[Bbar, s].{1, 0, 0} +
      Vector[Piota, s].Matrix[B, s].Matrix[B, s].Matrix[H3, s].{1, 0, 0}) +
   4 Vector[Piota, s].Matrix[B, s].Matrix[C2, s].{1, 0, 0} +
   4 Vector[Piota, s].Matrix[B, s].Matrix[B, s].Matrix[Abar, s].{1, 0, 0} +
   5 (Vector[hII, s].Matrix[Bbar, s].Matrix[Bbar, s].Vector[PNT, s] +
      Vector[Piota, s].Matrix[B, s].Matrix[B, s].
       (Matrix[H3, s].Vector[PNT, s] + Matrix[Ai, s].Vector[h, s])) +
   5 Vector[Piota, s].Matrix[B, s].Matrix[C2, s].Vector[PNT, s] +
   5 Vector[Piota, s].Matrix[B, s].Matrix[B, s].Matrix[Abar, s].Vector[PNT, s];
 , {s, {i, o}}]
```

In[212]:=
```
Do[
 Do[
  v[j] = EigenVectorBForPIota[j, s];
  vb[j] = EigenVectorBbar[j, s];
  e[j] = EigenValueBForPIota[j, s];
```

```
 eb[j] = EigenValueBbar[j, s];
 , {j, {1, 2, 3}}];
Bound[Xi, iota, EvenTail, Delta, 0, s] =
 Vector[hII, s].Sum[vb[j] * eb[j]^3 (2/(1 - eb[j]^2)^2 + 3/(1 - eb[j]^2)), {j, {1, 2, 3}}] +
  Sum[e[j]^3 (2/(1 - e[j]^2)^2 + 3/(1 - e[j]^2)) * v[j], {j, {1, 2, 3}}].
    (Matrix[H3, s].Vector[P, s] + Matrix[Ai, s].Vector[h, s]) +
  Sum[v[j] (2 e[j])/(1 - e[j]^2)^2, {j, {1, 2, 3}}].
    (Matrix[C2, s].Matrix[Bbar, s] + Matrix[B, s].Matrix[C1, s]).
   Sum[vb[j] 1/(1 - eb[j]^2), {j, {1, 2, 3}}] +
  Sum[v[j] e[j]/(1 - e[j]^2)^2, {j, {1, 2, 3}}].
    (Matrix[C2, s].Matrix[Bbar, s] + Matrix[B, s].Matrix[C1, s]).
   Sum[vb[j] (2/(1 - eb[j]^2)^2 + 1/(1 - eb[j]^2)), {j, {1, 2, 3}}];
Bound[Xi, iota, EvenTail, Delta, ei, s] =
 Vector[hII, s].Sum[vb[j] * eb[j]^3 (2/(1 - eb[j]^2)^2 + 4/(1 - eb[j]^2)), {j, {1, 2, 3}}] +
  Sum[e[j]^3 (2/(1 - e[j]^2)^2 + 4/(1 - e[j]^2)) * v[j], {j, {1, 2, 3}}].
    (Matrix[H3, s].Vector[P, s] + Matrix[Ai, s].Vector[h, s]) +
  Sum[v[j] (2 e[j])/(1 - e[j]^2)^2, {j, {1, 2, 3}}].
    (Matrix[C2, s].Matrix[Bbar, s] + Matrix[B, s].Matrix[C1, s]).
   Sum[vb[j] 1/(1 - eb[j]^2), {j, {1, 2, 3}}] +
  Sum[v[j] e[j]/(1 - e[j]^2)^2, {j, {1, 2, 3}}].
    (Matrix[C2, s].Matrix[Bbar, s] + Matrix[B, s].Matrix[C1, s]).
   Sum[vb[j] (2/(1 - eb[j]^2)^2 + 2/(1 - eb[j]^2)), {j, {1, 2, 3}}] +
  Sum[e[j]^3 (2/(1 - e[j]^2)^2 + 4/(1 - e[j]^2)) * v[j], {j, {1, 2, 3}}].Matrix[Ai, s].
   Vector[P, s];

Bound[Xi, iota, OddTail, Delta, 0, s] =
 Vector[hII, s].Sum[vb[j] * eb[j]^4 (2/(1 - eb[j]^2)^2 + 4/(1 - eb[j]^2)), {j, {1, 2, 3}}] +
```

```
   Sum[e[j]^4 (2/(1 - e[j]^2)^2 + 4/(1 - e[j]^2)) * v[j], {j, {1, 2, 3}}].
     (Matrix[H3, s].Vector[P, s] + Matrix[Ai, s].Vector[h, s]) +
   Sum[v[j] (2 e[j])/(1 - e[j]^2)^2, {j, {1, 2, 3}}].Matrix[C2, s].
    Sum[vb[j] 1/(1 - eb[j]^2), {j, {1, 2, 3}}] +
   Sum[v[j] e[j]/(1 - e[j]^2), {j, {1, 2, 3}}].Matrix[C2, s].
    Sum[vb[j] 2/(1 - eb[j]^2)^2, {j, {1, 2, 3}}] -
   4 Vector[Piota, s].Matrix[B, s].Matrix[C2, s].Vector[P, s] +
   Sum[v[j] (2 e[j]^2)/(1 - e[j]^2)^2, {j, {1, 2, 3}}].Matrix[C1, s].
    Sum[vb[j] 1/(1 - eb[j]^2), {j, {1, 2, 3}}] +
   Sum[v[j] (2 e[j]^2)/(1 - e[j]^2), {j, {1, 2, 3}}].Matrix[C1, s].
    Sum[vb[j] (2/(1 - eb[j]^2)^2 + 2/(1 - eb[j]^2)), {j, {1, 2, 3}}];
 Bound[Xi, iota, OddTail, Delta, ei, s] =
  Vector[hII, s].Sum[vb[j] * eb[j]^4 (2/(1 - eb[j]^2)^2 + 5/(1 - eb[j]^2)), {j, {1, 2, 3}}] +
   Sum[e[j]^4 (2/(1 - e[j]^2)^2 + 5/(1 - e[j]^2)) * v[j], {j, {1, 2, 3}}].
     (Matrix[H3, s].Vector[P, s] + Matrix[Ai, s].Vector[h, s]) +
   Sum[v[j] (2 e[j])/(1 - e[j]^2)^2, {j, {1, 2, 3}}].Matrix[C2, s].
    Sum[vb[j] 1/(1 - eb[j]^2), {j, {1, 2, 3}}] +
   Sum[v[j] e[j]/(1 - e[j]^2), {j, {1, 2, 3}}].Matrix[C2, s].
    Sum[vb[j] 2/(1 - eb[j]^2)^2, {j, {1, 2, 3}}] +
   Sum[v[j] (2 e[j]^2)/(1 - e[j]^2)^2, {j, {1, 2, 3}}].Matrix[C1, s].
    Sum[vb[j] 1/(1 - eb[j]^2), {j, {1, 2, 3}}] +
   Sum[v[j] (2 e[j]^2)/(1 - e[j]^2), {j, {1, 2, 3}}].Matrix[C1, s].
```

```
    Sum[vb[j] (2/(1 - eb[j]^2)^2 + 3/(1 - eb[j]^2)), {j, {1, 2, 3}}] +
   Sum[e[j]^3 (2/(1 - e[j]^2)^2 + 5/(1 - e[j]^2)) * v[j], {j, {1, 2, 3}}].Matrix[Ai, s].
    Vector[P, s];
 , {s, {i, o}}]
```

For Sum over all N

```
In[213]:= Do[
  Do[
   Bound[Xi, a, Even, Delta, 0, s] =
    Sum[Bound[Xi, a, t, Delta, 0, s], {t, {0, 2, EvenTail}}];
   Bound[Xi, a, Odd, Delta, 0, s] =
    Sum[Bound[Xi, a, t, Delta, 0, s], {t, {1, 3, OddTail}}];
   Bound[Xi, a, Absolut, Delta, 0, s] =
    Sum[Bound[Xi, a, t, Delta, 0, s], {t, {Odd, Even}}];
   , {a, {normal, iota}}];

  Bound[Xi, iota, Even, Delta, ei, s] =
   Sum[Bound[Xi, iota, t, Delta, ei, s], {t, {0, 2, EvenTail}}];
  Bound[Xi, iota, Odd, Delta, ei, s] =
   Sum[Bound[Xi, iota, t, Delta, ei, s], {t, {1, 3, OddTail}}];
  Bound[Xi, iota, Absolut, Delta, ei, s] =
   Sum[Bound[Xi, iota, t, Delta, ei, s], {t, {Odd, Even}}];
  , {s, {i, o}}]
```

Computation of constants of Proposition 3.3.1

In this section we perform the computations of Section 3.4. to obtain the constances stated in Proposition 3.3.1:

$$\begin{gathered}\sum_x |F(x)| \leq K_F = \text{Bound[KF]} \\ \text{Bound[KPhi, 1]} = \underline{K}_\Phi \leq \hat{\Phi}(0) \leq \overline{K}_\Phi = \text{Bound[KPhi, 2]} \\ \text{Bound[KPhiabs, 1]} = \underline{K}_{|\Phi|} \leq \sum_x \left|\Phi(x)\right| \leq \overline{K}_{|\Phi|} = \text{Bound[KPhiabs, 2]} \\ \sum_{x\neq 0} |\Phi_z(x)| \leq K_{|\Phi|} = \text{Bound[KPhiWithoutZero]}\end{gathered} \tag{1}$$

$$\begin{gathered}\sum_x F(x)[1-\cos(k\,x)] \geq K_{\text{Lower}}\left[1-\hat{D}(k)\right] \\ \sum_x \left|F(x)\right|[1-\cos(k\,x)] \leq K_{\Delta \text{F}}\left[1-\hat{D}(k)\right] \\ \sum_x \left|\Phi_z(x)\right|[1-\cos(k\,x)] \leq K_{\Delta\Phi}\left[1-\hat{D}(k)\right]\end{gathered} \tag{2}$$

Bound on absolute value K_F and K_Φ

In[214]:=
```
Do[
  alpha[s] = z[s] rho[s];
  baralpha[s] = z[s];
  Bound[KPsi, s] = rho[s] + (2 d - 2)/(2 d) Bound[Xi, normal, Absolut, s];
  Bound[KF, s] =
   (2 d baralpha[s]) / (1 - alpha[s] - (2 d - 2) alpha[s] Bound[Xi, iota, Absolut, s])
    Bound[KPsi, s];
  Bound[KPhiup, s] = 1 + Bound[Xi, normal, Even, s] +
    Bound[Xi, iota, Absolut, s] Bound[KF, s];
  Bound[KPhidown, s] = 1 - Bound[Xi, normal, Odd, s] -
    Bound[Xi, iota, Absolut, s] Bound[KF, s];
  Bound[KPhiabsup, s] = 1 + Bound[Xi, normal, Absolut, s] +
    Bound[Xi, iota, Absolut, s] Bound[KF, s];
  Bound[KPhiabsdown, s] = 1 - Bound[Xi, normal, Absolut, s] -
    Bound[Xi, iota, Absolut, s] Bound[KF, s];
  Bound[KPhiWithoutZero, s] =
   Bound[Xi, normal, Absolut, s] + Bound[Xi, iota, Absolut, s] Bound[KF, s];
  , {s, {i, o}}]
```

Bounds on differences

As next we implement the computation of Section 3.4.3. First the differences of $F_1$and Φ_1, lines (3.4.26), (3.4.27), (3.4.29)

In[215]:=
```
Do[
  Bound[DifferencefF, Part1, Lower, s] =
   Min[(2 d baralpha[i])/(1 - alpha[i]^2), (2 d baralpha[s])/(1 - alpha[s]^2)]
    (rho[Lower, s] - Bound[Xi, normal, Odd, Delta, 0, s] - Bound[Xi, normal, Odd, s] -
      alpha[o] Bound[Xi, normal, Even, Delta, 0, s]);
  Bound[DifferencefF, Part1, Absolut, s] =
   Max[(2 d baralpha[i])/(1 - alpha[i]^2), (2 d baralpha[s])/(1 - alpha[s]^2)]
    (rho[s] + (1 + alpha[s]) Bound[Xi, normal, Absolut, Delta, 0, s] +
      Bound[Xi, normal, Absolut, s]);
  Bound[KDeltaPhi, Part1, s] =
   Bound[Xi, normal, Absolut, Delta, 0, s] +
    baralpha[s]/(1 - alpha[s]^2)
     (2 d Bound[Xi, normal, Absolut, Delta, 0, s] Bound[Xi, iota, Absolut, s] +
       (1 + Bound[Xi, normal, Absolut, s]) Bound[Xi, iota, Absolut, Delta, ei, s] +
       2 d alpha[s] Bound[Xi, normal, Absolut, Delta, 0, s] Bound[Xi, iota, Absolut, s] +
       alpha[s] (1 + Bound[Xi, normal, Absolut, s])
        Bound[Xi, iota, Absolut, Delta, 0, s]);
  , {s, {i, o}}]
```

Then the differences of $F_2$and Φ_2: For the bound in (3.4.30), (3.4.32) and (3.4.35)

```
In[216]:= Do[
  Bound[DifferencefF, Part2, Lower, s] =
   -(2 d baralpha[s])^2/(1 - alpha[s]^2)
     (Bound[Xi, normal, Odd, Delta, 0, s] Bound[Xi, iota, Odd, s] +
       Bound[Xi, normal, Even, Delta, 0, s] Bound[Xi, iota, Even, s])
    - (2 d baralpha[s]^2)/(1 - alpha[s]^2) (rho[s] + (2 d - 2)/(2 d) Bound[Xi, normal, Even, s])
     ( Bound[Xi, iota, Even, Delta, ei, s] + 2 d Bound[Xi, iota, Even, s] +
       alpha[s]^2 Bound[Xi, iota, Even, Delta, 0, s] +
       alpha[s] (Bound[Xi, iota, Odd, Delta, ei, s] + Bound[Xi, iota, Odd, Delta, 0, s] +
          2 d Bound[Xi, iota, Odd, s])) -
    (2 d baralpha[s]^2)/(1 - alpha[s]^2) ((2 d - 2)/(2 d) Bound[Xi, normal, Odd, s])
     ( Bound[Xi, iota, Odd, Delta, ei, s] + 2 d Bound[Xi, iota, Odd, s] +
       alpha[s]^2 Bound[Xi, iota, Odd, Delta, 0, s] +
       alpha[s] (Bound[Xi, iota, Even, Delta, ei, s] +
          Bound[Xi, iota, Even, Delta, 0, s] + 2 d Bound[Xi, iota, Even, s]));
  Bound[DifferencefF, Part2, Absolut, s] =
   (2 d baralpha[s])^2/(1 - alpha[s]^2) Bound[Xi, normal, Absolut, Delta, 0, s]
     Bound[Xi, iota, Absolut, s]
    + (2 d baralpha[s]^2)/((1 - alpha[s]^2) (1 + alpha[s])) (rho[s] + (2 d - 2)/(2 d) Bound[Xi, normal, Absolut, s])
     ( Bound[Xi, iota, Absolut, Delta, ei, s] + 2 d Bound[Xi, iota, Absolut, s] +
       alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s]);
  Bound[KDeltaPhi, Part2, s] =
   (2 d baralpha[s]^2)/(1 - alpha[s]^2)
    (2 d Bound[Xi, normal, Absolut, Delta, 0, s] Bound[Xi, iota, Absolut, s]^2 +
      2 (1 + Bound[Xi, normal, Absolut, s])
       Bound[Xi, iota, Absolut, s]/(1 + alpha[s])
        (Bound[Xi, iota, Absolut, Delta, ei, s] +
          alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s]));
  , {s, {i, o}}]
```

Finally, we compute the differences of F_3 and Φ_3, lines (4.4.37) and (4.4.38)

In[217]:=
```
Do[
 tmp = 1 / (1 - (2 d alpha[s] Bound[Xi, iota, Absolut, s]) / (1 - alpha[s]));
 Bound[DifferencefF, Part3, Absolut, s] =
  Bound[Xi, normal, Absolut, Delta, 0, s] (2 d baralpha[s]) / (1 - alpha[s])^3
    (2 d alpha[s] Bound[Xi, iota, Absolut, s])^2 tmp +
   (1 + Bound[Xi, normal, Absolut, s]) (baralpha[s] (2 d alpha[s])^2) / ((1 - alpha[s])^2 (1 - alpha[s]^2))
    Bound[Xi, iota, Absolut, s] tmp^2
    (Bound[Xi, iota, Absolut, Delta, ei, s] +
      alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s]) +
   (1 + Bound[Xi, normal, Absolut, s]) (baralpha[s] (2 d alpha[s])^2) / ((1 - alpha[s])^2 (1 - alpha[s]^2))
    Bound[Xi, iota, Absolut, s] tmp
    (Bound[Xi, iota, Absolut, Delta, ei, s] +
      alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s] +
      2 d Bound[Xi, iota, Absolut, s]);

 Bound[DifferencefF, Part3, Lower, s] = -Bound[DifferencefF, Part3, Absolut, s];
 Bound[KDeltaPhi, Part3, s] =
  Bound[Xi, normal, Absolut, Delta, 0, s] (2 d baralpha[s] Bound[Xi, iota, Absolut, s]) / (1 - alpha[s])^3
    (2 d alpha[s] Bound[Xi, iota, Absolut, s])^2 tmp +
   (1 + Bound[Xi, normal, Absolut, s])
    (baralpha[s] (2 d alpha[s] Bound[Xi, iota, Absolut, s])^2) /
     ((1 - alpha[s])^2 (1 - alpha[s]^2)) (tmp^2 + tmp)
    1 / (1 + alpha[s]) (Bound[Xi, iota, Absolut, Delta, ei, s] +
       alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s]);

 Bound[KDeltaFLower, s] =
  1 / (Bound[DifferencefF, Part1, Lower, s] + Bound[DifferencefF, Part2, Lower, s] +
     Bound[DifferencefF, Part3, Lower, s]);
 Bound[KDeltaF, s] = Bound[DifferencefF, Part1, Absolut, s] +
   Bound[DifferencefF, Part2, Absolut, s] + Bound[DifferencefF, Part3, Absolut, s];
 Bound[KDeltaPhi, s] = Bound[KDeltaPhi, Part1, s] + Bound[KDeltaPhi, Part2, s] +
   Bound[KDeltaPhi, Part3, s];
 Clear[tmp];
 , {s, {i, o}}]
```

Computation of additial bounds of Assumption 3.5.3

Now we compute bounds on α_F, α_Φ, $R_F$and R_Φ as given in Section 3.5.3 and 4.3.9. We follow the structure of Section 4.3.9. We know that $z \geq z_I = e^{-1}/(2\,d-1)$ and by simple combinatorics that

$$g_z^i \geq 1 + (2\,d-1)\,z_I(1 + (2\,d-1)\,z_I) + (2\,d-1)\,(2\,d-2)\,\frac{z_I^2}{2}$$

In[218]:=
```
z[i] = 1/((2 d - 1) Exp[1]);
z[o] = Gamma1/(2 d - 1);(* Upper bound on z and thereby also on z_c *)
```

To rewrite Φ as in (3.5.31)-(3.5.33) we extract from $\Xi^{(0)}$ and $\Xi^{(1)}$ the nearest neighbor contribtion. For the implementation we split $\Xi^{(0)}$, $\Xi^{(1)}$ and $\Xi^{(0),i}$ as follows

In[220]:=
```
Bound[Xi, normalalphaPhi, 0, o] = Bound[ClosedRepLoop, 4, o];
Bound[Xi, normalRPhi, 0, o] = Bound[ClosedRepBubble, 4, o];

Bound[Xi, normalalphaPhi, 1, o] = Bound[ClosedRepLoop, 4, o];
Bound[Xi, normalRPhi, 1, o] =
  Vector[P, o].Matrix[Abar, o].Vector[P, o] - {1, 0, 0}.Matrix[Abar, o].{1, 0, 0} +
   Bound[ClosedRepBubble, 4, o];

Bound[Xi, iotaalphaPhi, 0, o] = Bound[G3i, o];
Bound[Xi, iotaRPhi, 0, o] = Bound[G3i, o] (Bound[P, 0, o] - 1);
```

We use these quantities to define the bounds

In[226]:=
```
ap = Max[Bound[Xi, normalalphaPhi, 0, o],
   Bound[Xi, normalalphaPhi, 1, o] + (2 d z[o])/(1 - z[o]^2) Bound[Xi, iotaalphaPhi, 0, o]];
Bound[Phi2Phi3, Absolut, 0, o] =
  ((2 d z[o])/(1 - z[o]))^2 (d - 1)/d Bound[KPsi, o] Bound[Xi, iota, Absolut, o]^2
   1/(1 - z[o] - (2 d - 2) z[o] Bound[Xi, iota, Absolut, o]);(*compare with (3.4.11)*)

bRp = Bound[Xi, normalRPhi, 0, o] + Bound[Xi, normalRPhi, 1, o] +
   Sum[Bound[Xi, normal, t, o], {t, {2, 3, EvenTail, OddTail}}] +
   (2 d z[o])/(1 - z[o]^2)
    (Bound[Xi, iotaRPhi, 0, o] + Sum[Bound[Xi, normal, t, o],
       {t, {1, 2, 3, EvenTail, OddTail}}]) +
   (2 d z[o])/(1 - z[o]^2) Bound[Xi, normal, Absolut, o] Bound[Xi, iota, Absolut, o] +
   Bound[Phi2Phi3, Absolut, 0, o];
```

see (3.5.32)-(3.5.33).

To compute $\Phi(0) - \Phi(k)$ we compute the remainder term for the difference $\Phi_1(0) - \Phi_1(k)$:

In[229]:=
```
Bound[Xi, normalRPhi, 0, Delta, 0, o] = Bound[WeightedClosedBubble, 2, o];
Bound[Xi, normalRPhi, 1, Delta, 0, o] =
  Bound[Xi, normal, 1, Delta, 0, o] - Bound[H2, 0, 0, o] +
   Bound[WeightedClosedBubble, 2, o];

Bound[Xi, iotaRPhi, 0, DeltaPhi, 0, o] =
  2 d Bound[G3i, o] (Bound[P, 0, o] - 1) +
   2 d Bound[G3i, o] Bound[WeightedClosedBubble, 2, o];
Bound[Xi, iotaRPhi, 0, DeltaPhi, ei, o] =
  2 d Bound[G3i, o] Bound[WeightedClosedBubble, 2, o];


Bound[Xi, iotaRPhi, Absolut, o] =
  Bound[Xi, iota, Absolut, o] - Bound[Xi, iota, 0, o] + Bound[Xi, iotaRPhi, 0, o];
Bound[Xi, iotaRPhi, Absolut, Delta, 0, o] =
  Bound[Xi, iota, Absolut, Delta, 0, o] - Bound[Xi, iota, 0, Delta, 0, o] +
   Bound[Xi, iotaRPhi, 0, DeltaPhi, 0, o];
Bound[Xi, iotaRPhi, Absolut, Delta, ei, o] =
  Bound[Xi, iota, Absolut, Delta, ei, o] - Bound[Xi, iota, 0, Delta, ei, o] +
   Bound[Xi, iotaRPhi, 0, DeltaPhi, ei, o];
```

Then, we use these differences and add the already computed differences $\Phi_2(0) - \Phi_2(k)$ and $\Phi_3(0) - \Phi_3(k)$:

In[236]:=
```
bRpDelta = Bound[Xi, normalRPhi, 0, Delta, 0, o] +
   Bound[Xi, normalRPhi, 1, Delta, 0, o] +
   Sum[Bound[Xi, normal, t, Delta, 0, o], {t, {2, 3, EvenTail, OddTail}}] +
     z[o]
   ―――――――――
   1 - z[o]^2
    (2 d Bound[Xi, normal, Absolut, Delta, 0, o] Bound[Xi, iotaRPhi, Absolut, o] +
      (1 + Bound[Xi, normal, Absolut, o]) Bound[Xi, iotaRPhi, Absolut, Delta, ei, o] +
      2 d z[o] Bound[Xi, normal, Absolut, Delta, 0, o] Bound[Xi, iotaRPhi, Absolut, o] +
      z[o] (1 + Bound[Xi, normal, Absolut, o])
       Bound[Xi, iotaRPhi, Absolut, Delta, 0, o]) +
     z[o]
   ―――――――――
   1 - z[o]^2
   (2 d z[o] Bound[Xi, normal, Absolut, Delta, 0, o] Bound[Xi, iotaalphaPhi, 0, o] / (2 d) +
      z[o] Bound[Xi, normal, Absolut, o] Bound[Xi, iotaalphaPhi, 0, o]) +
   Bound[KDeltaPhi, Part2, o] + Bound[KDeltaPhi, Part3, o];
```

The first term corresponds are contributions to Ξ_z that have not been extracted. The second term bound $\sum_i \Psi^{(0),i}(0)\,(\hat{\Xi}^i(0) - \Xi^i(e_i))$. The third term bounds $\sum_i \left(\hat{\Psi}^{(0),i}(0) - \Psi^{(0),i}(0)\right) \Xi^i(e_i)\big)$. In last term bound all remainder term the contribution of $\Phi_2$and Φ_3.

For the rewrite of $1 - F(k)$ we require the following quantities:

```
In[237]:= Psi1eiekLower = c3i z[i]^4 + 20 (2 d - 2) z[i]^6; (*Ψ^{1,k}(e1+e2)*)
Psi0e1Lower = (2 d - 2) z[i]^4 + (2 (2 d - 2 ) + 2 d - 3 + 3 (2 d - 2) (2 d - 4) + (2 d - 4)^2 ) z[i]^6;
(*Ψ^{0,k}(e1)*)
Psi0e2Lower = (2 d - 3) z[i]^4 + (2 (2 d - 2 ) + 2 d - 3 + 3 (2 d - 2) (2 d - 4) + (2 d - 4)^2) z[i]^6;
(*Ψ^{0,k}(e2)*)

Psi0eiek = z[o]^2 (z[o]^2 + c4ik z[o]^4 + c6ik z[o]^6 + Bound[G8ik, o]) +
    c4ik/2 z[o]^4 (c6ik z[o]^6 + Bound[G8ik, o]) + Bound[G6ik, o] Bound[G6ik, o];
 (*Ψ^{0,k}(ei+ek)*)
Psi1e2 = Bound[ClosedRepBubble, 4, o]/(2 d z[o]) Bound[G1i, o] +
    Bound[ClosedRepBubble, 3, o]/(2 d z[o]) Bound[OpenRepBubble, 2, o] Max[1, Bound[G1, o]];
Psi1e1 = Psi1e2;
```

We use as bound on the absolute value of α_F(3.5.39) the following

```
In[243]:= af = 2 d z[o]/(1 - z[o]^2) + 2 d z[o] (2 d - 2)/(1 - z[o]^2) (Psi0eiek - Psi1eiekLower) +
    2 d z[o] z[o]/(1 - z[o]^2) ((2 d - 2) (Psi1e2 - Psi0e2Lower) + (Psi1e1 - Psi0e1Lower));
```

For the computation of $F(0) - F(k)$ we use the following values

```
In[244]:= Bound[Xi, normalRf, 0, Delta, 0, o] = Bound[WeightedClosedBubble, 2, o];
Bound[Xi, normalRf, 1, Delta, 0, o] =
  Bound[Xi, normal, 1, Delta, 0, o] - Bound[H2, 0, 0, o] +
   Bound[WeightedClosedBubble, 2, o];
Bound[Xi, normalRf, 0, o] = Bound[ClosedRepBubble, 4, o];
Bound[Xi, normalRf, 1, o] =
 Bound[Xi, normal, 1, Delta, 0, o] - Bound[H2, 0, 0, o] +
  Bound[WeightedClosedBubble, 2, o];
Bound[Xi, normalRf, Absolut, o] =
 Bound[Xi, normal, Absolut, o] +
  Sum[Bound[Xi, normalRf, t, o] - Bound[Xi, normal, t, o], {t, 0, 1}];
Bound[Xi, normalRf, Absolut, Delta, 0, o] =
  Bound[Xi, normal, Absolut, Delta, 0, o] +
   Sum[Bound[Xi, normalRf, t, Delta, 0, o] - Bound[Xi, normal, t, Delta, 0, o],
    {t, 0, 1}];
```

to create the bounds

```
In[249]:= bRf = 2 d z[o]/(1 - z[o]) (2 d - 2)/(2 d) Bound[Xi, normalRf, Absolut, o] +
    2 d z[o]/(1 - z[o]^2) Bound[KPsi, o] ((2 d-2)/(2 d) (2 d z[o])/(1-z[o]) Bound[Xi, iota, Absolut, o])/(1 - (2 d z[o])/(1-z[o]) Bound[Xi, iota, Absolut, o]);
bRfDelta =
  2 d z[o]/(1 - z[o]^2) (Bound[Xi, normalRf, Absolut, Delta, 0, o] +
      Bound[Xi, normalRf, Absolut, o] +
      z[o] Bound[Xi, normalRf, Absolut, Delta, 0, o]) +
   Bound[DifferencefF, Part2, Absolut, o] + Bound[DifferencefF, Part3, Absolut, o];
```

Check of the sufficient condition

Now we can compute whether Q(γ,Γ,z) is satisfied, see Definition 3.5.6.

In[251]:=
```
Do[
 NoBLEBoundF1[s] = (1 + (2 d-2)/(2 d-1) Gamma1 Bound[Xi, iota, Even, s]) / (rho[s] - (2 d-2)/(2 d) Bound[Xi, normal, Odd, s]);
 NoBLEBoundF2[s] = (2 d - 2)/(2 d - 1) Bound[KPhiabsup, s] Bound[KDeltaFLower, s];
 , {s, {i, o}}]
```

In[252]:=
```
BoundFThree[0, 0]
```

Out[252]=
```
BoundFThree[0, 0]
```

We finally check

In[253]:=
```
Do[
 Succes[f1, s] = NoBLEBoundF1[s] < Gamma1;
 Succes[f2, s] = NoBLEBoundF2[s] < Gamma2;
 Succes[s] = Succes[f1, s] && Succes[f2, s];
 , {s, {i, o}}]
```

Further, we need the constants for the improvement of $\bar{f}_{3,n,l}$ and $\bar{f}_{4,n,l}$:

In[254]:=
```
BoundFFour[n_, l_] := BoundFFourBar[n, l, (2 d - 2)/(2 d - 1) Gamma2, twodgz[s], 1, af,
   ap, bRf, bRp, bRfDelta, bRpDelta, Bound[KDeltaFLower, o]];
BoundFThree[n_, l_] := BoundFThreeBar[n, l, (2 d - 2)/(2 d - 1) Gamma2, twodgz[s], 1, af,
   ap, bRf, bRp, bRfDelta, bRpDelta, Bound[KDeltaFLower, o]];
```

In[256]:=
```
Do[
 SuccesFThree[t + 2] = BoundFThree[1, t] < GammaThree[1, t];
 , {t, 0, 3}]
SuccesFThree[1] = BoundFThree[0, 0] < GammaThree[0, 0];
Succes[f3bar] = SuccesFThree[1] && SuccesFThree[2] && SuccesFThree[3] &&
   SuccesFThree[4] && SuccesFThree[5];

Succes[f4bar] = BoundFFour[1, 4] < GammaFour[1, 4];

Succes[overall] = Succes[i] && Succes[o] && Succes[f3bar] && Succes[f4bar];
```

Result

The overall result

The statement that the bootstrap was succesful is

In[261]:=
```
Succes[overall]
```

Out[261]=
```
True
```

If this succedes than the analysis of Section 3.5 can be used to proved mean-field behavoior for Percolation.

Assuming the bootstrap was succesful we prove the following bounds: The infrared bound in (3.3.12) and (3.3.13) hold with

In[262]:=
$$\frac{2\,d-2}{2\,d-1}$$ Gamma2(* ≥ $G_z(k)\left[1-\hat{D}(k)\right]$ *)
Max[Bound[KDeltaFLower, o], 1]
(* Nominator in (4.3.13) *)

Out[262]= 1.00136

Out[263]= 1.0594

Further, we have proven that $g_{z_c} z_c$ is smaller than

In[264]:=
$$\frac{1}{2\,d-1}$$ Gamma1

Out[264]= 0.0360131

The improvement of bounds

In[265]:=
```
bubbles = {Graphics[{Green, Disk[{0, 0}, 0.8]}, ImageSize -> 30],
   Graphics[{Red, Disk[{0, 0}, 0.8]}, ImageSize -> 40]};
tableClassicCheck = {{Bounds, Init - f1, Init - f2, f1, f2, f̄4,1,4},
   {Gamma, Gamma1, Gamma2, Gamma1, Gamma2, GammaFour[1, 4] },
   {Bounds, NoBLEBoundF1[i], NoBLEBoundF2[i], NoBLEBoundF1[o], NoBLEBoundF2[o],
    GammaFour[1, 4]}, { check,
    If[NoBLEBoundF1[i] < Gamma1, bubbles[[1]], bubbles[[2]]],
    If[NoBLEBoundF2[i] < Gamma2, bubbles[[1]], bubbles[[2]]],
    If[NoBLEBoundF1[o] < Gamma1, bubbles[[1]], bubbles[[2]]],
    If[NoBLEBoundF2[o] < Gamma2, bubbles[[1]], bubbles[[2]]],
    If[BoundFThree[0, 0] < GammaThree[0, 0], bubbles[[1]], bubbles[[2]]]}};
tableClassicCheckFthree = {{Bounds, "(0,0)", "(1,0)", "(1,1)", "(1,2)", "(1,3)"},
   {Gamma, GammaThree[0, 0], GammaThree[1, 0], GammaThree[1, 1], GammaThree[1, 2],
    GammaThree[1, 3] }, {Bounds, BoundFThree[0, 0], BoundFThree[1, 0],
    BoundFThree[1, 1], BoundFThree[1, 2], BoundFThree[1, 3]}, { check,
    If[SuccesFThree[1], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[2], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[3], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[4], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[5], bubbles[[1]], bubbles[[2]]]}};
Labeled[Grid[tableClassicCheckFthree, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True, 5 → True}},
   ItemStyle → {1 → Bold, 1 → Bold},
   Background → {Automatic, Automatic, {{3, 3}, {2, 9}} → GrayLevel[0.9]}],
  Style["Result for f3 in  Dimension " Text[d], Bold], Top] // Text

Labeled[Grid[tableClassicCheck, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {Automatic, Automatic, {{2, 2}, {2, 15}} → GrayLevel[0.7]}],
  Style["Result for Dimension " Text[d], Bold], Top] // Text
```

Result for f_3 in Dimension 15

Out[268]=

Bounds	**(0,0)**	**(1,0)**	**(1,1)**	**(1,2)**	**(1,3)**
Gamma	0.14058	0.197926	0.047	0.02301	0.00837828
Bounds	0.140492	0.197802	0.0469632	0.0229971	0.00837381
check					

Result for Dimension 15

Out[269]=

Bounds	**Init – f_1**	**Init – f_2**	**f_1**	**f_2**	**$\bar{f}_{4,1,4}$**
Gamma	1.04438	1.03712	1.04438	1.03712	0.00837843
Bounds	1.0429	0.999453	1.04438	1.03708	0.00837843
check					

Semi-automate proceedure to find appropiate value for the constants $\Gamma_i$and c_i.

Initially we guess a good value for the constant and make a first computation. Then we deactivate there initial definition in the top of the document and use the code below. We recompile the document a number of times and hope that the algorithm below converges to a fixed point.

In[270]:=
```
{d, Gamma1, Gamma2, GammaThree[0, 0], GammaThree[1, 0], GammaThree[1, 1],
 GammaThree[1, 2], GammaThree[1, 3], GammaFour[1, 4]}
Gamma1 = NoBLEBoundF1[o] + 0.0000001;
Gamma2 = NoBLEBoundF2[o] + 0.0000001;
Do[
 GammaThree[1, t] = BoundFThree[1, t] + 0.0000001;
 , {t, 0, 3}]
GammaFour[1, 4] = BoundFFour[1, 4] + 0.0000001;
GammaThree[0, 0] = BoundFThree[0, 0] + 0.0000001;
{d, Gamma1, Gamma2, GammaThree[0, 0], GammaThree[1, 0], GammaThree[1, 1],
 GammaThree[1, 2], GammaThree[1, 3], GammaFour[1, 4]}
```

Out[270]=
```
{15, 1.04438, 1.03712, 0.14058, 0.197926, 0.047, 0.02301, 0.00837828, 0.00837843}
```

Out[276]=
```
{15, 1.04438, 1.03708, 0.140491, 0.197798,
 0.0469623, 0.0229966, 0.00837375, 0.00837375}
```

```
In[277]:= Do[
  MethodeFourTable[s] = {{Quantity, ΞZero, ΞOne, ΞTwo, ΞThree, ΞEvenTail, ΞOddTail},
     {Text[Bound for Ξ̂], Bound[Xi, normal, 0, s], Bound[Xi, normal, 1, s] ,
      Bound[Xi, normal, 2, s], Bound[Xi, normal, 3, s], Bound[Xi, normal, EvenTail, s],
      Bound[Xi, normal, OddTail, s]},
     {Text[Bound for Ξ̂ι], Bound[Xi, iota, 0, s], Bound[Xi, iota, 1, s] ,
      Bound[Xi, iota, 2, s], Bound[Xi, iota, 3, s], Bound[Xi, iota, EvenTail, s],
      Bound[Xi, iota, OddTail, s]},
     {Text[Ξ̂ (1 - cos (kx))], Bound[Xi, normal, 0, Delta, 0, s],
      Bound[Xi, normal, 1, Delta, 0, s], Bound[Xi, normal, 2, Delta, 0, s],
      Bound[Xi, normal, 3, Delta, 0, s], Bound[Xi, normal, EvenTail, Delta, 0, s],
      Bound[Xi, normal, OddTail, Delta, 0, s]},
     {Text[ Ξι (1 - cos (kx))], Bound[Xi, iota, 0, Delta, 0, s],
      Bound[Xi, iota, 1, Delta, 0, s], Bound[Xi, iota, 2, Delta, 0, s],
      Bound[Xi, iota, 3, Delta, 0, s], Bound[Xi, iota, EvenTail, Delta, 0, s],
      Bound[Xi, iota, OddTail, Delta, 0, s]},
     {Text[Ξι (1 - cos (k (x - eι)))] , Bound[Xi, iota, 0, Delta, ei, s],
      Bound[Xi, iota, 1, Delta, ei, s] , Bound[Xi, iota, 2, Delta, ei, s],
      Bound[Xi, iota, 3, Delta, ei, s], Bound[Xi, iota, EvenTail, Delta, ei, s],
      Bound[Xi, iota, OddTail, Delta, ei, s]}};
  , {s, {i, o}}]
MethodeFourTablePart1 = {{Quantity, KF, KPhi, DELTAFLower, DELTAFAbsolut, DELTAPhi},
   {Bound for , Bound[KF, o], Bound[KPhiup, o], Bound[KDeltaFLower, o],
    Bound[KDeltaF, o], Bound[KDeltaPhi, o]}};
MethodeFourTablePart2 =
  {{Quantity, DELTAFLower, 2, 3, DELTAFAbsolut, 2, 3, DELTAPhi, 2, 3},
   {Bound for , Bound[DifferencefF, Part1, Lower, o],
    Bound[DifferencefF, Part2, Lower, o], Bound[DifferencefF, Part3, Lower, o],
    Bound[DifferencefF, Part1, Absolut, o], Bound[DifferencefF, Part2, Absolut, o],
    Bound[DifferencefF, Part3, Absolut, o], Bound[KDeltaPhi, Part1, o],
    Bound[KDeltaPhi, Part2, o], Bound[KDeltaPhi, Part3, o]}};

Labeled[Grid[MethodeFourTable[i], Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Bound on coefficients at zI in Dimension " Text[d], Bold], Top] // Text
Labeled[Grid[MethodeFourTable[o], Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Bound on coefficients in Dimension " Text[d], Bold], Top] // Text

Labeled[Grid[MethodeFourTablePart1, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Bound on the constants of Proposition 3.3.1 in Dimension " Text[d], Bold],
  Top] // Text
Labeled[Grid[MethodeFourTablePart2, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Bound on the constants of Proposition 3.3.1 in Dimension " Text[d], Bold],
  Top] // Text
```

Out[280]=

Bound on coefficients at z_I in Dimension 15

Quantity	Ξ^{Zero}	Ξ^{One}	Ξ^{Two}	Ξ^{Three}	$\Xi^{EvenTail}$	$\Xi^{OddTail}$
Bound for $\hat{\Xi}$	0.00383349	0.00411683	0.000613996	0.0000283919	1.53115×10^{-6}	7.99155×10^{-8}
Bound for $\hat{\Xi}^{\iota}$	0.00148224	0.000557842	0.0000303571	1.56923×10^{-6}	8.27819×10^{-8}	4.33866×10^{-9}
$(1-\cos kx)\,\hat{\Xi}$	0.0107432	0.0132189	0.00667152	0.000508696	0.0000702638	0.000960162
$(1-\cos kx)\,\Xi^{\iota}$	0.0449273	0.027891	0.00768348	0.000674259	0.0000572954	0.0000524875
$\Xi^{\iota}\,(1-\cos k\,(x-e_{\iota}))$	0.000460034	0.0111558	0.00330044	0.000364155	0.0000832831	0.000237813

Out[281]=

Bound on coefficients in Dimension 15

Quantity	Ξ^{Zero}	Ξ^{One}	Ξ^{Two}	Ξ^{Three}	$\Xi^{EvenTail}$	$\Xi^{OddTail}$
Bound for $\hat{\Xi}$	0.00482755	0.00525289	0.000968097	0.0000562547	3.79988×10^{-6}	2.48452×10^{-7}
Bound for $\hat{\Xi}^{\iota}$	0.00174532	0.000740587	0.0000499481	3.24393×10^{-6}	2.14467×10^{-7}	1.40782×10^{-8}
$(1-\cos kx)\,\hat{\Xi}$	0.0191173	0.0315013	0.0343843	0.00316348	0.000552816	0.00606361
$(1-\cos kx)\,\Xi^{\iota}$	0.0533224	0.0474513	0.0271497	0.00325267	0.000456591	0.000457869
$\Xi^{\iota}\,(1-\cos k\,(x-e_{\iota}))$	0.00096296	0.0252337	0.0151863	0.00215152	0.000585332	0.00166917

Out[282]=

Bound on the constants of Proposition 3.3.1 in Dimension 15

Quantity	KF	KPhi	DELTAFLower	DELTAFAbsolut	DELTAPhi
Bound for	1.09358	1.00858	1.0594	1.16601	0.0968992

Out[283]=

Bound on the constants of Proposition 3.3.1 in Dimension 15

Quantity	DELTA⋱ FLow⋱ er	2	3	DELTA⋱ FAbs⋱ olut	2	3	DELTA⋱ Phi	2	3
Bound for	0.946911	−0.0029⋱ 645⋱ 5	−0.0000⋱ 199⋱ 426	1.16107	0.00492⋱ 463	0.00001⋱ 99426	0.09688⋱ 87	0.00001⋱ 04531	2.8797×10^{-8}