Computation for the lace expansion for lattice animals

Analysis of Section 3.5-3.6

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Abstract

In this file we performs the numerical part of the analysis of the non-backtracking lace expansion for lattice animals. All references in this version of the notebook will be to the PhD thesis of the author.

This file is accomanied by another notebook -SRW_Computations- where an number of simple random walks are computed. The user should first open that file, choose a dimension and execute all lines of the file. Then he is expects to choose constance Γ_i in this file. After choosing these quantities the used should select the menu item Evaluate-> Evaluate Notebook. In a table at the end of this document the result of the computations are shown. There it can be see whether the bootstrap with the given parameters and therefore the analysis was succesful. The computation of the -SRW_Computations- file are independent of the values Γ_i, so that the need to compute the SRW-integral once when we start the program and whenever we change the dimension.

We compute bound on the two-point function and repulsive diagrams (Section 5.1.2). We use these bounds to compute the Diagramatic bounds derived in Section 4.4. and compute the bounds used for the Analysis in Section 3.5.

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Input

We try to perform the bootstrap for the following values of Γ_i. So that the values of f_1, f_2,$\bar{f}_{3,n,l}$ and $\bar{f}_{4,n,l}$ ars small then:

In[3718]:=
```
Gamma1 = 1.0534841;
Gamma2 = 1.1688239;

GammaFour[1, 4] = 0.003472;
GammaFour[2, 4] = 0.0070783636;

GammaThree[1, 0] = 0.16725101;
GammaThree[1, 1] = 0.03088468;
GammaThree[1, 2] = 0.012046;
GammaThree[1, 3] = 0.003472;
GammaThree[2, 0] = 0.2482931;
GammaThree[2, 1] = 0.05456131;
GammaThree[2, 2] = 0.0215825;
GammaThree[2, 3] = 0.0070784;
```

In[3730]:=

The bootstraps succes in dimension 21 with the constants (* deactivated*)

```
(*Gamma1=1.0534841;
Gamma2=1.1688239;

GammaFour[1,4]=0.003472;
GammaFour[2,4]=0.0070783636;

GammaThree[1,0]=0.16725101;
GammaThree[1,1]=0.03088468;
GammaThree[1,2]=0.012046;
GammaThree[1,3]=0.003472;
GammaThree[2,0]=0.2482931;
GammaThree[2,1]=0.05456131;
GammaThree[2,2]=0.0215825;
GammaThree[2,3]=0.0070784;*)
```

Bound on the two-point function and on repulsive diagrams

Definition of Constants

We define the constants for two setting s: we use s=i for bound on $z = z_I$ and s=o for bound on $z \epsilon (z_I, z_c)$: For $z = z_I$, we use the following relations, that are proven in Section 3.6.2.

$$\begin{aligned} z_I &= \frac{1}{(2d-1)e} \\ g_{z_I} &\leq e + \frac{e-1}{2d-1} \\ g^i_{z_I} &\leq 1 + (g_{z_I} - 1)\frac{2d-1}{2d} \leq e \\ G_z(x) &\leq g^i_{z_I} B_{z_I\, g^i_{z_I}}(x) \leq g^i_{z_I} \frac{2d-2}{2d-1} C(x) \\ \tilde{G}_z(x) &\leq B_{z_I\, g^i_{z_I}}(x) \leq \frac{2d-2}{2d-1} C(x). \end{aligned} \tag{1}$$

For z ϵ (z_I, z_c) we know that

$$\begin{aligned} 2\,d\,z\,g^i_z &< 2\,d\,g_z\,z < \Gamma_1 \\ g_z &< e\,\Gamma_1 \text{ for all } z \geq \frac{1}{(2d-1)e} \\ g^i_z &< 1 + (g_z - 1)\frac{2d-1}{2d} \end{aligned} \tag{2}$$

In the following we implement the basic quantities for the status i at z_I and the status o for z in (z_I, z_c), which will allows us to implement the bound for both (s=i,o) with the same code

In[3732]:=
```
g[i] = Exp[1] + (Exp[1] - 1)/(2 d - 1);
g[o] = Exp[1] * Gamma1;
gj[i] = Exp[1];
gj[o] = 1 + (g[o] - 1) * (2 d - 1)/(2 d);
rho[i] = gj[i]/g[i];
rho[o] = gj[o]/g[o];
twodgz[i] = 2 d 1/((2 d - 1) Exp[1]) g[i];
twodgz[o] = (2 d)/(2 d - 1) Gamma1;
twodgjz[i] = (2 d)/(2 d - 1);
twodgjz[o] = (2 d)/(2 d - 1) Gamma1;
gjz[i] = 1/(2 d - 1);
gjz[o] = 1/(2 d - 1) Gamma1;

VarGamma1[i] = (1 + (1 - Exp[-1])/(2 d - 1));
VarGamma1[o] = Gamma1;
VarGamma2[i] = rho[i] * (2 d - 2)/(2 d - 1); (*G_z(x) ≤g_z B_{zg_z}(x) ≤  g_z (2d-2)/(2d-1) C(x)*)
VarGamma2[o] = Gamma2 * (2 d - 2)/(2 d - 1); (*Ĝ_z(k)≤ ConstantGvsC Ĉ(k), follows from f2*)
VarGamma3[i] = 1; (*We bound a weighted line by replacing the tree two-
  point function with a normal on*)
VarGamma3[o] = Gamma3;(*Ĝ_z(k)≤ Ĉ(k)*)
Varc1[o] = c1; Varc2[o] = c2; Varc3[o] = c3; Varc4[o] = c4;
Varc1[i] = 0; Varc2[i] = 0.5; Varc3[i] = 0; Varc4[i] = 4;
```

Further, we define variables to save the number of short SAWs, as given in Section 5.1.3

In[3752]:=
```
c2ik = 2; (*c_2(e_1+e_2)*)
c4ik = 4 (2 d - 3) + 2 (2 d - 4);(*c_4(e_1+e_2)*)
c6ik = 16 + 84 (2 d - 4) + 36 (2 d - 4) (2 d - 6) + 6 d c3i;(*c_4(e_1+e_2)*)

c3i = (2 d - 2);(*c_3(e_1)*)
c5i = (3 (2 d - 2 ) + 4 (2 d - 2) (2 d - 4) );(*c_5(e_1)*)
c7i = (14 (2 d - 2 ) + 62 (2 d - 2) (2 d - 4) + 27 (2 d - 2) (2 d - 4) (2 d - 6) ) + 8 d c3i + 4 d c5i;
(*c_7(e_1)*)
```

Bounds on two-point function

We compute bounds as explained in Section 5.1.2. We begin by computing $G_{n,z}(e_1)$ as in (5.1.22)-(5.1.24)

In[3758]:=
```
Do[
  Bound[tG7i, s] = c7i (gjz[s])^7 + (twodgjz[s])^9 * VarGamma2[s] * I110; (* G_{7,z}(e_1)*)
  Bound[tG5i, s] = c5i (gjz[s])^5 + Bound[tG7i, s]; (* G_{5,z}(e_1)*)
  Bound[tG3i, s] = c3i (gjz[s])^3 + Bound[tG5i, s]; (* G_{3,z}(e_1)*)
  Bound[tG1i, s] = gjz[s] + Bound[tG3i, s];(* G_{1,z}(e_1)*)
  , {s, {i, o}}]
```

Then we compute $G_{n,z}(e_1 + e_2)$ and $G_{4,z}(2\, e_2)$, see (5.1.25)-(5.1.26) :

In[3759]:=
```
Do[
  Bound[tG8ik, s] = d/(d - 1) (twodgjz[o])^8 VarGamma2[s] I110; (*G_8(e_1+e_2)*)
  Bound[tG6ik, s] = c6ik gjz[s]^6 + VarGamma2[s] I18; (*G_6(e_1+e_2)*)
  Bound[tG4ik, s] = Bound[tG6ik, s] + (c4ik - 2 (2 d - 3) ) gjz[s]^4; (*G_4^1(e_1+e_2)*)
  Bound[tG2ik, s] = Bound[tG4ik, s] + (c2ik - 1) gjz[s]^2;(*G_2^1(e_1+e_2)*)
  (*Bound[tG4ii,s]=(2d+2)gjz[s]^4+Bound[tG6,s];(* Bound for sup_x G_4^1(2 e_1)*)*)
  , {s, {i, o}}]
```

We compute the supreme of the two-point function as given in (5.1.27)-(5.1.31):

In[3760]:=
```
Do[
  Bound[tG6, s] = Max[c6ik gjz[s]^6, c7i gjz[s]^7] + (twodgjz[s])^8 VarGamma2[s] I18;
  (* Bound for sup_x G_6(x)*)
  Bound[tG4, s] = Max[c4ik gjz[s]^4, c5i gjz[s]^5] + Bound[tG6, s];
  (* Bound for sup_x G_4(x)*)
  Bound[tG2, s] = Max[c2ik gjz[s]^2, c3i gjz[s]^3] + Bound[tG4, s];
  (* Bound for sup_x G_2(x)*)
  Bound[tG1, s] = Max[Bound[tG1i, s], Bound[tG2, s]];(* Bound for sup_x G_1(x)*)
  , {s, {i, o}}]
```

Closed repulsive diagrams

We define the bounds on the closed repulsive diagrams as desribed in Section 5.1.2 in (5.1.34)-(5.1.36). The bound does only depent on the total number of steps and the number of tw-point functions involved. It does not depent on the indivitual length of the pieces m_1, m_2, ... and of the orientation of the arrows.

```
In[3761]:= Do[
  Bound[ClosedRepLoop, 4, s] = twodgjz[s] Bound[tG3i, s];
  Bound[ClosedRepBubble, 4, s] =
   gjz[s]^4 (2 d c3i) + 3 gjz[s]^6 (2 d c5i) + 5 gjz[s]^8 (2 d c7i) +
    6 twodgjz[s]^10 VarGamma2[s] I110 + twodgjz[s]^10 VarGamma2[s]^2 I210;
  Bound[ClosedRepBubble, 3, s] =
   Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s];
  Bound[ClosedRepTriangle, 4, s] =
   (4 + 1 - 4) (4 + 2 - 4)/2 gjz[s]^4 (2 d c3i) + (6 + 1 - 4) (6 + 2 - 4)/2 gjz[s]^6 (2 d c5i) +
    (8 + 1 - 4) (8 + 2 - 4)/2 gjz[s]^8 (2 d c7i) +
    (10 - 4) (9 - 4)/2 twodgjz[s]^10 VarGamma2[s] I110 + 6 twodgjz[s]^10 VarGamma2[s]^2 I210 +
    twodgjz[s]^10 VarGamma2[s]^3 I310;
  Bound[ClosedRepTriangle, 3, s] =
   Bound[ClosedRepTriangle, 4, s] + Bound[ClosedRepBubble, 3, s];
  Bound[ClosedRepTriangle, 2, s] =
   Bound[ClosedRepTriangle, 3, s] + Bound[ClosedRepBubble, 3, s] +
    Bound[ClosedRepLoop, 4, s];
  Bound[ClosedRepSquare, 4, s] =
   gjz[s]^4 (2 d c3i) + 10 gjz[s]^6 (2 d c5i) + 35 gjz[s]^8 (2 d c7i) +
    84 twodgjz[s]^10 VarGamma2[s] I110 + (10 - 4) (9 - 4)/2 twodgjz[s]^10 VarGamma2[s]^2 I210 +
    6 twodgjz[s]^10 VarGamma2[s]^3 I310 + twodgjz[s]^10 VarGamma2[s]^4 I410;
  , {s, {i, o}}]
```

Open repulsive diagrams

Then we define the bound on the open repulsive diagrams as in (5.1.38):

In[3762]:=
```
Do[
 Bound[OpenRepBubble, 2, s] =
  Max[c2ik gjz[s]^2 + 3 c4ik gjz[s]^4 + 5 c6ik gjz[s]^6,
    2 c3i gjz[s]^3 + 4 c5i gjz[s]^5 + 6 c7i gjz[s]^7] + 6 twodgjz[s]^8 VarGamma2[s] I18 +
   twodgjz[s]^8 VarGamma2[s]^2 I28;
 Bound[OpenRepBubble, 3, s] =
  Max[2 c4ik gjz[s]^4 + 4 c6ik gjz[s]^6, c3i gjz[s]^3 + 3 c5i gjz[s]^5 + 5 c7i gjz[s]^7] +
   5 twodgjz[s]^8 VarGamma2[s] I18 + twodgjz[s]^8 VarGamma2[s]^2 I28;
 Bound[OpenRepBubble, 4, s] =
  Max[c4ik gjz[s]^4 + 3 c6ik gjz[s]^6, c5i gjz[s]^5 + 4 c7i gjz[s]^7] +
   4 twodgjz[s]^8 VarGamma2[s] I18 + twodgjz[s]^8 VarGamma2[s]^2 I28;
 Bound[OpenRepTriangle, 1, s] =
  Max[3 c2ik gjz[s]^2 + 10 c4ik gjz[s]^4 + 21 c6ik gjz[s]^6,
    gjz[s] + 6 c3i gjz[s]^3 + 15 c5i gjz[s]^5 + 28 c7i gjz[s]^7] +
   (8 - 1) (7 - 1)/2 twodgjz[s]^8 VarGamma2[s] I18 + 7 twodgjz[s]^8 VarGamma2[s]^2 I28 +
   twodgjz[s]^8 VarGamma2[s]^3 I38;
 Bound[OpenRepTriangle, 2, s] =
  Max[c2ik gjz[s]^2 + 6 c4ik gjz[s]^4 + 15 c6ik gjz[s]^6,
    3 c3i gjz[s]^3 + 10 c5i gjz[s]^5 + 21 c7i gjz[s]^7] +
   (8 - 2) (7 - 2)/2 twodgjz[s]^8 VarGamma2[s] I18 + 6 twodgjz[s]^8 VarGamma2[s]^2 I28 +
   twodgjz[s]^8 VarGamma2[s]^3 I38;
 Bound[OpenRepTriangle, 3, s] =
  Max[3 c4ik gjz[s]^4 + 10 c6ik gjz[s]^6, c3i gjz[s]^3 + 6 c5i gjz[s]^5 + 15 c7i gjz[s]^7] +
   (8 - 3) (7 - 3)/2 twodgjz[s]^8 VarGamma2[s] I18 + 5 twodgjz[s]^8 VarGamma2[s]^2 I28 +
   twodgjz[s]^8 VarGamma2[s]^3 I38;
 Bound[OpenRepSquare, 3, s] =
  Max[4 c4ik gjz[s]^4 + 20 c6ik gjz[s]^6, c3i gjz[s]^3 + 10 c5i gjz[s]^5 + 35 c7i gjz[s]^7] +
   56 twodgjz[s]^8 VarGamma2[s] I18 + (8 - 3) (7 - 3)/2 twodgjz[s]^8 VarGamma2[s]^2 I28 +
   5 twodgjz[s]^8 VarGamma2[s]^3 I38 + twodgjz[s]^8 VarGamma2[s]^4 I48;
 , {s, {i, o}}]
```

Weighted Diagrams

First we define weighted diagams as explained in Section 5.1., e.g. (5.1.19) and (5.1.42)-(5.1.49) we derive weighted closed diagrams

First we define the bound on the weighted diagams in the same format as we used the in the implementation for the analysis of Section 3.3.

For $z = z_i$ we use the bound (3.6.20) :

In[3763]:=
```
Bound[WeightedClosedBubble, 4, i] = twodgjz[i]^4 BoundFFourBarInitial[1, 4, rho[i]];
Bound[WeightedClosedTriangle, 4, i] = twodgjz[i]^4 BoundFFourBarInitial[2, 4, rho[i]];

Do[
 Bound[WeightedOpenBubble, t, i] = twodgjz[i]^t BoundFThreeBarInitial[1, t, rho[i]];
 Bound[WeightedOpenTriangle, t, i] = twodgjz[i]^t BoundFThreeBarInitial[2, t, rho[i]];
 , {t, 0, 3}]
```

Then we use $f_{3,n,l}$ and $f_{4,n,l}$ that gives us direct bound for the weighted diagrams for $z \in (z_i, z_c)$:

```
In[3766]:= Bound[WeightedClosedBubble, 4, o] = twodgjz[o]^4 GammaFour[1, 4];
Bound[WeightedClosedTriangle, 4, o] = twodgjz[o]^4 GammaFour[2, 4];

Do[
 Bound[WeightedOpenBubble, t, o] = twodgjz[i]^t GammaThree[1, t];
 Bound[WeightedOpenTriangle, t, o] = twodgjz[i]^t GammaThree[2, t];
 , {t, 0, 3}]
```

Then we use the idea explained in (5.1.42)-(5.1.49) to obtain

```
In[3769]:= Do[
 Bound[WeightedClosedBubble, 2, s] =
  Bound[WeightedClosedBubble, 4, s] + 8 d gjz[s]^2 (gjz[s]^4 (2 d - 2) * 2 + Bound[tG6, s]) +
   8 d gjz[s]^2 (2 d - 2) (gjz[s]^2 + Bound[tG4ik, s]) +
   2 d gjz[s]^3 (Bound[tG3i, s] + 3 (2 d - 2) d/(d - 1) d/(d - 2) twodgjz[s]^3 VarGamma2[s] I16 +
      5 (2 d - 2) d/(d - 1) Bound[tG6, s] +
      9 (gjz[s]^2 + 3 (2 d - 2) gjz[s]^5 + Bound[tG6, s]));
 Bound[WeightedClosedBubble, 0, s] =
  Bound[WeightedClosedBubble, 2, s] + twodgjz[s] Bound[tG3i, s];

 Bound[WeightedClosedTriangle, 2, s] =
  2 Bound[WeightedClosedBubble, 4, s] + Bound[WeightedClosedTriangle, 4, s] +
   8 d gjz[s]^2 (gjz[s]^4 (2 d - 2) * 2 + Bound[tG6, s]) +
   8 d gjz[s]^2 (2 d - 2) (gjz[s]^2 + Bound[tG4ik, s]) +
   4 d gjz[s]^3 (Bound[tG3i, s] + 3 (2 d - 2) d/(d - 1) d/(d - 2) twodgjz[s]^3 VarGamma2[s] I16 +
      5 (2 d - 2) d/(d - 1) Bound[tG6, s] + 9 (gjz[s]^2 + 3 (2 d - 2) gjz[s]^5 + Bound[tG6, s]));
 , {s, {i, o}}]
```

We define the elements as given in (4.3.31)-(4.3.37) and (4.3.49)-(4.3.51)

```
In[3770]:= Do[
  Bound[Delta, I, 0, s] = 2 twodgjz[s] Bound[tG3i, s] +
    2 Bound[WeightedClosedBubble, 2, s] + Bound[WeightedClosedTriangle, 2, s];
  Bound[Delta, I, 1, s] = Bound[tG3i, s] + Bound[WeightedClosedBubble, 2, s]/twodgjz[s] +
    Bound[WeightedClosedTriangle, 2, s]/twodgjz[s];
  Bound[Delta, I, 2, s] = Bound[WeightedOpenTriangle, 0, s];
  Bound[Delta, I, 3, s] = 2 Bound[WeightedOpenBubble, 2, s]/twodgjz[s] +
    2 Bound[WeightedOpenTriangle, 3, s]/twodgjz[s];
  Bound[Delta, I, 4, s] = Bound[WeightedOpenBubble, 1, s] +
    Bound[WeightedOpenBubble, 2, s]/twodgjz[s] + 2 Bound[WeightedOpenTriangle, 2, s]/twodgjz[s];
  Bound[Delta, I, 5, s] = Bound[WeightedClosedTriangle, 2, s]/twodgjz[s]^2;
  Bound[Delta, I, 6, s] = 2 Bound[WeightedClosedBubble, 2, s] +
    Bound[WeightedClosedTriangle, 4, s];

  Bound[Delta, II, 0, s] =
   twodgjz[s] Bound[tG3i, s] + Bound[WeightedClosedBubble, 2, s] +
    Bound[WeightedClosedTriangle, 2, s];
  Bound[Delta, II, 1, s] = Bound[WeightedClosedTriangle, 2, s]/twodgjz[s];
  Bound[Delta, II, 2, s] = Bound[WeightedOpenTriangle, 1, s];
  , {s, {i, o}}]

In[3771]:= s = o;
  Bound[Delta, I, 0, s]
  Bound[Delta, I, 1, s]
  Bound[Delta, I, 2, s]
  Bound[Delta, I, 3, s]
  Bound[Delta, I, 4, s]
  Clear[s];

Out[3772]= 0.0408287

Out[3773]= 0.0293978

Out[3774]= 0.248293

Out[3775]= 0.0375282

Out[3776]= 0.0853242
```

Bound on the coefficients

Definition of Initial Pieces (P , P')

To implement the bound of N≥1 we define the boulding matricies given in (4.3.27)-(4.3.53):

```
In[3778]:= Do[
  (* definition of first peices, inpendent of fiota*)
```

```
Bound[P1, 0, 0, s] =
  (3 Bound[ClosedRepLoop, 4, s] + 3 Bound[ClosedRepBubble, 4, s] +
    Bound[ClosedRepTriangle, 4, s]);
Bound[P1, 0, 1, s] =
  (2 Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s] +
    Bound[ClosedRepTriangle, 4, s]);
Bound[P1, 0, 2, s] =
  (Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s] +
    Bound[ClosedRepTriangle, 4, s] + Bound[ClosedRepSquare, 4, s]);
Bound[P1, 0, -1, s] =
  (Bound[ClosedRepLoop, 4, s] + 2 Bound[ClosedRepBubble, 4, s] +
    Bound[ClosedRepTriangle, 4, s]);
Bound[P1, 0, -2, s] =
  (Bound[ClosedRepTriangle, 4, s] + Bound[ClosedRepSquare, 4, s]);;
(* definition of first peices, first step of the backbone goes to e_i*)
Do[Bound[P1, IotaStep, t, s] = Bound[P1, 0, t, s], {t, {-2, -1, 0, 1, 2}}];
Bound[P1, IotaRib, 0, s] =
  2 d (Bound[tG1i, s] Bound[P1, 0, 0, s] + Bound[ClosedRepBubble, 4, s] +
     2 Bound[tG2, s] Bound[ClosedRepLoop, 4, s] +
     Bound[tG1, s] Bound[ClosedRepBubble, 4, s] +
     Bound[tG3i, s] 1/twodgjz[s] Bound[ClosedRepBubble, 3, s] +
     (Bound[ClosedRepBubble, 4, s])/twodgjz[s] Bound[OpenRepBubble, 2, s] (*c≠x,
     d_omega=1*) + 2 Bound[OpenRepBubble, 3, s]
      (Bound[tG1i, s] + 1/twodgjz[s] Bound[ClosedRepBubble, 3, s]) (*c≠x,
     d_omega≥ 2;u≠v=x*) + Bound[OpenRepTriangle, 3, s]
      (Bound[tG1i, s] + 1/twodgjz[s] Bound[ClosedRepBubble, 3, s]) (*u≠v≠x*));
(*definition of the first piece if e_i is somewhere on the first rib*)
Bound[P1, IotaRib, 1, s] =
  2 d (Bound[tG1i, s] Bound[P1, 0, 1, s] + Bound[ClosedRepBubble, 4, s] +
     Bound[ClosedRepBubble, 4, s] + Bound[ClosedRepTriangle, 4, s] +
     1/twodgjz[s] Bound[ClosedRepBubble, 3, s] Bound[OpenRepBubble, 2, s] +
     (Bound[tG1i, s] + 1/twodgjz[s] Bound[ClosedRepBubble, 3, s])
      Bound[OpenRepTriangle, 3, s]);
Bound[P1, IotaRib, 2, s] =
  2 d (Bound[tG1i, s] Bound[P1, 0, 2, s] +
     (Bound[tG1i, s] + 1/twodgjz[s] Bound[ClosedRepBubble, 3, s])
      Bound[OpenRepSquare, 3, s]);
Bound[P1, IotaRib, -2, s] =
```

```
 2 d (Bound[tG1i, s] Bound[P1, 0, -2, s] +
     (Bound[tG1i, s] + 1/twodgjz[s] Bound[ClosedRepBubble, 3, s])
      Bound[OpenRepSquare, 3, s]);
Bound[P1, IotaRib, -1, s] =
 2 d (Bound[tG1i, s] Bound[P1, 0, -1, s] +
     Bound[tG1, s] (Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s] +
        Bound[ClosedRepTriangle, 4, s]) +
     (Bound[tG1i, s] + 1/twodgjz[s] Bound[ClosedRepBubble, 3, s])
      Bound[OpenRepTriangle, 3, s]);
, {s, {i, o}}]
```

Definition of Initial Pieces (Q, Q^{ι})

Now we define the vectors for the first diagram of the coefficients (4.4.6)-(4.4.23)

In[3779]:=
```
Do[
 (* definition of first pieces, inpendent of fiota*)
 Bound[Q0, 0, s] = Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s];
 Bound[Q0, 1, s] = Bound[tG3i, s] + Bound[ClosedRepBubble, 4, s];
 Bound[Q0, 2, s] = Bound[ClosedRepBubble, 4, s] + Bound[ClosedRepTriangle, 4, s];

 Do[ (*Line (4.4.10) *)
  Bound[Q1, t, s] = Bound[P1, 0, t, s] +
     Sum[Bound[Q0, r, s] Bound[A, -r, t, s], {r, 0, 2}];
  , {t, {-2, -1, 0, 1, 2}}];
 Bound[Q0, sausage, p1, 0, s] =
  Bound[tG3i, s] Bound[tG1i, s] + Bound[ClosedRepBubble, 3, s] Bound[tG1i, s] / gjz[s] +
   Bound[tG2, s] (Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 3, s]) +
   Bound[tG1, s] (Bound[ClosedRepBubble, 4, s] + Bound[ClosedRepTriangle, 4, s]);
 Bound[Q0, sausage, 0, s] =
  Bound[tG1i, s] Bound[Q0, 0, s] + Bound[Q0, sausage, p1, 0, s];
 Bound[Q0, sausage, p1, 1, s] =
  Bound[tG1i, s] Bound[Q0, 1, s] + Bound[ClosedRepBubble, 3, s] Bound[tG1i, s] / gjz[s] +
    (Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 3, s]) Bound[tG1i, s] / gjz[s] +
   Bound[tG2, s]
    (Bound[ClosedRepBubble, 3, s] + Bound[ClosedRepLoop, 4, s] +
      Bound[ClosedRepBubble, 3, s]) +
   Bound[tG1, s] (Bound[ClosedRepTriangle, 4, s] + Bound[ClosedRepTriangle, 3, s]);
 Bound[Q0, sausage, 1, s] =
  Bound[tG1i, s] Bound[Q0, 1, s] + Bound[Q0, sausage, p1, 1, s];
 Bound[Q0, sausage, p1, 2, s] =
  (2 Bound[ClosedRepTriangle, 3, s] + Bound[ClosedRepBubble, 3, s]) Bound[tG1i, s] / gjz[s] +
   Bound[tG1, s] Bound[ClosedRepSquare, 4, s] +
   Bound[tG2, s] Bound[ClosedRepTriangle, 3, s] +
   Bound[tG1, s] Bound[ClosedRepSquare, 4, s];
 Bound[Q0, sausage, 2, s] =
  Bound[tG1i, s] Bound[Q0, 2, s] + Bound[Q0, sausage, p1, 2, s];
 Do[ (*Line (4.4.14) *)
  Bound[Q1, sausage, t, s] = Sum[Bound[Q0, sausage, r, s] Bound[A, -r, t, s],
    {r, 0, 2}];
  (* improvement ot (4.4.22)*)
  Bound[Q1, step, t, s] =
   (gjz[s]^2 + Bound[tG4ik, s]) Bound[tG4ik, s]
     (Bound[A, 0, t, s] + 2 Bound[A, -1, t, s]) +
    Bound[ClosedRepBubble, 4, s] Bound[tG2ik, s] / gjz[s] Bound[A, -2, t, s];
  (* improvement ot (4.4.23)*)
  Bound[Q1, iota, t, s] = Bound[P1, IotaStep, t, s] + Bound[P1, IotaRib, t, s] +
    Bound[Q1, sausage, t, s] + Bound[Q1, step, t, s];
  , {t, {-2, -1, 0, 1, 2}}];
 , {s, {i, o}}]
```

Definition of the intermediate pieces (A $\overline{A}$)

In[3780]:=
```
Do[
 (*definition of itermediate peices, where one shared edges is counted*)
 Do[Bound[A, 0, a, s] = Bound[P1, 0, a, s], {a, {-2, -1, 0, 1, 2}}];
 Bound[A, 1, 0, s] =
  Bound[tG3i, s] +
   1/twodgjz[s] (Bound[ClosedRepLoop, 4, s] + 2 Bound[ClosedRepBubble, 4, s] +
      Bound[ClosedRepTriangle, 4, s]);
 Bound[A, 2, 0, s] = Bound[OpenRepBubble, 2, s] + Bound[OpenRepTriangle, 2, s];

 Bound[A, 1, 1, s] =
  1/twodgjz[s] (Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s] +
     Bound[ClosedRepTriangle, 4, s]);
 Bound[A, 1, 2, s] = Bound[ClosedRepTriangle, 4, s]/twodgjz[s];
 Bound[A, 2, 1, s] = Bound[OpenRepTriangle, 2, s];
 Bound[A, 2, 2, s] = Bound[OpenRepSquare, 3, s];

 Do[
  Bound[A, -t1, 0, s] = Bound[A, t1, 0, s];
  , {t1, 1, 2}];
 Clear[t1];

 (*definition of itermediate peices, where both shared edges are not counted*)
 Do[Bound[Abar, a, 0, s] = gj[s] Bound[A, a, 0, s], {a, {-2, -1, 0, 1, 2}}];
 (*(4.3.25)*)
 Do[Bound[Abar, 0, a, s] = gj[s] Bound[A, a, 0, s], {a, {-2, -1, 0, 1, 2}}];
 (*(4.3.26)*)
 Bound[Abar, 1, 1, s] =
  gj[s]
   1/(twodgjz[s])^2 (Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s] +
      Bound[ClosedRepTriangle, 4, s]);
 Bound[Abar, 1, 2, s] = gj[s] Bound[OpenRepTriangle, 2, s]/twodgjz[s];
 Bound[Abar, 2, 1, s] = Bound[Abar, 1, 2, s];
 Bound[Abar, 2, 2, s] = gj[s] Bound[OpenRepTriangle, 1, s];


 (*Using Symmetrie we define the other ones*)
 Do[Do[
   Do[
    Bound[t, a, -b, s] = Bound[t, a, b, s];
    Bound[t, -a, b, s] = Bound[t, a, b, s];
    Bound[t, -a, -b, s] = Bound[t, a, b, s];
    , {a, {1, 2}}], {b, {1, 2}}], {t, {A, Abar}}];
 (*the more complex pieces*)
 , {s, {i, o}}];
```

Definition of the Delta entries

In[3781]:=
```
Do[
 Bound[Delta, start, 2, s] = Bound[Delta, I, 2, s];
 Bound[Delta, start, 1, s] = Bound[Delta, I, 1, s];
 Bound[Delta, start, 0, s] = Bound[Delta, I, 0, s];
 Bound[Delta, start, -1, s] = Bound[Delta, I, 1, s];
 Bound[Delta, start, -2, s] = Bound[Delta, I, 2, s];

 Bound[Delta, end, 2, s] = Bound[Delta, I, 4, s];
 Bound[Delta, end, 1, s] = Bound[Delta, I, 3, s];
 Bound[Delta, end, 0, s] = Bound[Delta, I, 0, s];
 Bound[Delta, end, -1, s] = Bound[Delta, I, 1, s];
 Bound[Delta, end, -2, s] = Bound[Delta, I, 2, s];

 Bound[Delta, 0, 0, s] = Bound[Delta, I, 0, s];
 Bound[Delta, 1, 0, s] = Bound[Delta, I, 3, s];
 Bound[Delta, 0, -1, s] = Bound[Delta, I, 1, s];
 Bound[Delta, -1, 0, s] = Bound[Delta, I, 1, s];
 Bound[Delta, 0, 1, s] = Bound[Delta, I, 1, s];
 Bound[Delta, -1, 1, s] = Bound[Delta, I, 5, s];
 Bound[Delta, -1, -1, s] = Bound[Delta, I, 5, s];
 Bound[Delta, 1, 1, s] = Bound[Delta, I, 6, s];
 Bound[Delta, 1, -1, s] = Bound[Delta, I, 6, s];

 Do[
  Bound[Delta, -2, t, s] = Bound[Delta, I, 2, s];
  Bound[Delta, 2, t, s] = 2 Bound[Delta, I, 2, s];
  , {t, -2, 2}];
 Bound[Delta, -1, 2, s] = Bound[Delta, I, 2, s];
 Bound[Delta, -1, -2, s] = Bound[Delta, I, 2, s];
 Bound[Delta, 1, 2, s] = 2 Bound[Delta, I, 2, s];
 Bound[Delta, 1, -2, s] = 2 Bound[Delta, I, 2, s];
 Bound[Delta, 0, 2, s] = Bound[Delta, I, 2, s];
 Bound[Delta, 0, -2, s] = Bound[Delta, I, 2, s];
 Do[
  Bound[Delta, iotaI, t, s] = Bound[Delta, II, Abs[t], s] + Bound[Delta, I, Abs[t], s] +
    Bound[tG1i, s] Bound[Delta, 0, t, s] + Bound[tG3i, s] Bound[Delta, -1, t, s] +
    Bound[ClosedRepBubble, 4, s] / twodgjz[s] Bound[Delta, -2, t, s];
  Bound[Delta, iotaII, t, s] =
   Bound[Delta, II, Abs[t], s] + Bound[Delta, I, Abs[t], s] +
    Bound[tG1i, s] Bound[Delta, 0, t, s] + 2 Bound[tG3i, s] Bound[Delta, -1, t, s] +
    2 Bound[ClosedRepBubble, 4, s] / twodgjz[s] Bound[Delta, -2, t, s] +
    2 twodgjz[s] Bound[P1, IotaRib, t, s];
  , {t, -2, 2}];

 , {s, {i, o}}]
```

Then we defone the pieces for the non-trivial first triangle ($\underline{b}_0 \neq 0$) and the entries to bound $\Xi_z(0) - \Xi_z(k)$, see (4.4.25)-(4.4.28):

In[3782]:=
```
Do[
  Bound[hQZero, 0, s] = twodgjz[s] Bound[tG3i, s] + Bound[WeightedClosedBubble, 2, s];
  Bound[hQZero, 1, s] = Bound[tG3i, s] + Bound[WeightedClosedBubble, 2, s] / twodgjz[s];
  Bound[hQZero, 2, s] = Bound[WeightedClosedBubble, 0, s];
  Do[
   Bound[DeltaQ, start, t, s] = Bound[Delta, start, t, s] +
      2 (Sum[Bound[Q1, c, s] Bound[Delta, c, -2, s] +
          Bound[hQZero, c, s] Bound[A, c, t, s], {c, 0, 2}]);
   , {t, -2, 2}];
  , {s, {i, o}}];
```

Next we bound $\triangle^{iota,I}$ and $\triangle^{iota,II}$ as drawn Figure 4.18. The first and third images contribution also present for LT. The first bound the second diagram ($b_0 = (v, e_i)$) with $n \neq 0$ (step).

AS next we bound $\triangle^{iota,I}$ and $\triangle^{iota,I}$ as drawn in Figure 4.18. The first and third diagram are also present for LT and are bounded by Bound[Delta,iotaI or iotaI ,t,s]. The second diagram corresponds to $(b_0 = (v, e_1)$, with $v \neq 0$. we bound this diagram in Bound[DeltaQ,I/II,step,t,s]. Then be bound the fourth diagam (ribpart1). Then we declare the bounds on forth and sixth diagram (ribpart2)

In[3783]:=
```
Do[
 Do[
  Bound[DeltaQ, II, step, t, s] =
   Bound[tG2ik, s] Bound[tG3i, s] (Bound[Delta, -1, t, s] + 2 Bound[Delta, -2, t, s]) +
    Bound[tG2ik, s] Bound[OpenRepBubble, 4, s] Bound[Delta, -2, t, s];
  Bound[DeltaQ, I, step, t, s] =
   Bound[DeltaQ, II, step, t, s] +
    Bound[tG2ik, s] Bound[tG3i, s] (Bound[Abar, -1, t, s] + 2 Bound[Abar, -2, t, s]) +
    Bound[tG2ik, s] Bound[OpenRepBubble, 4, s] Bound[Abar, -2, t, s];
  Bound[DeltaQ, I, ribpart1, t, s] =
   2 d Bound[tG1i, s]
    (Bound[Delta, start, -t, s] +
      2 (Sum[Bound[P1, 0, -c, s] Bound[Delta, -c, t, s] +
          Bound[Delta, start, -c, s] Bound[A, -c, t, s], {c, 0, 2}])); (* u = 0 *)
  Bound[DeltaQ, II, ribpart1, t, s] =
   Bound[DeltaQ, I, ribpart1, t, s] +
    2 d Bound[tG1i, s] Sum[Bound[P1, 0, -c, s] Bound[A, -c, t, s], {c, 0, 2}];

  Bound[DeltaQ, I, ribpart2, t, s] =
   2 Sum[Bound[Q0, sausage, p1, c, s] Bound[Delta, -c, t, s], {c, 0, 2}] +
    2 Bound[WeightedOpenBubble, 0, s] Bound[OpenRepTriangle, 3, s]
     Sum[ Bound[A, -c, t, s], {c, 0, 2}] +
    2 Bound[WeightedClosedTriangle, 2, s] Bound[tG1, s]
     ( Bound[A, 0, t, s] + Bound[A, 2, t, s]) +
    2 Bound[WeightedClosedBubble, 2, s] Bound[tG1, s] Bound[A, 2, t, s] +
    4 Bound[WeightedOpenBubble, 0, s] Bound[ClosedRepTriangle, 3, s];
  Bound[DeltaQ, II, ribpart2, t, s] =
   Bound[DeltaQ, I, ribpart2, t, s] +
    Sum[Bound[Q0, sausage, r, s] Bound[A, -r, t, s], {r, 0, 2}]
  , {t, -2, 2}];
 Do[
  Bound[DeltaQ, I, t, s] = Bound[Delta, iotaI, t, s] +
    Sum[Bound[DeltaQ, I, a, t, s], {a, {step, ribpart1, ribpart2}}];
  Bound[DeltaQ, II, t, s] =
   Bound[Delta, iotaII, t, s] + Sum[Bound[DeltaQ, II, a, t, s],
     {a, {step, ribpart1, ribpart2}}];
  , {t, -2, 2}];
 , {s, {i, o}}]
```

Definition of the vectors and matices

We condition on the length of the backone and indentify whether the backbone is on the top or bottom of the diagram.

– the backbone in on the bottom, d (u, v) ≥ 2.
– the backbone in on the bottom, d (u, v) = 1.
u = v
– the backbone in on the top, d (u, v) = 1.
– the backbone in on the top, d (u, v) ≥ 2.

In[3784]:=
```
Do[
  VectorP1[normal, s] = Table[Bound[P1, 0, r - 3, s], {r, 1, 5}];
  VectorP1[iota, s] = Table[Bound[P1, IotaStep, r - 3, s] + Bound[P1, IotaRib, r - 3, s],
    {r, 1, 5}];
  VectorQ1[normal, s] = Table[Bound[Q1, r - 3, s], {r, 1, 5}];
  VectorQ1[iota, s] = Table[Bound[Q1, iota, r - 3, s], {r, 1, 5}];

  (*MatrixA[s]=Table[Max[Bound[A,r-3,t-3,s],Bound[A,t-3,r-3,s]],{t,1,5},
     {r,1,5}];*)
  MatrixA[s] = Table[Bound[A, r - 3, t - 3, s], {t, 1, 5}, {r, 1, 5}];
  MatrixAbar[s] = Table[Bound[Abar, r - 3, t - 3, s], {t, 1, 5}, {r, 1, 5}];
  VectorAbar[s] = Table[Bound[Abar, r - 3, 0, s], {r, 1, 5}];

  VectorDelta[startQ, s] = Table[Bound[DeltaQ, start, r - 3, s], {r, 1, 5}];
  VectorDelta[end, s] = Table[Bound[Delta, end, r - 3, s], {r, 1, 5}];
  VectorDelta[iotaIQ, s] = Table[Bound[DeltaQ, I, r - 3, s], {r, 1, 5}];
  VectorDelta[iotaIIQ, s] = Table[Bound[DeltaQ, II, r - 3, s], {r, 1, 5}];
  MatrixDelta[s] = Table[Bound[Delta, r - 3, t - 3, s], {t, 1, 5}, {r, 1, 5}];
  , {s, {i, o}}]
```

To compute the geometric sum over matricies we compute a representation of $P^{(1)}$ and $P^{(1)\iota}$ by eigenvalue of the matricies A:

In[3785]:=
```
Do[
  EigenA[s] = Eigensystem[Transpose[MatrixA[s]]];
  InverseProductP[normal, s] =
   Inverse[Transpose[EigenA[s][[2]]]].VectorP1[normal, s];
  InverseProductQ[normal, s] =
   Inverse[Transpose[EigenA[s][[2]]]].VectorQ1[normal, s];
  InverseProductQ[iota, s] = Inverse[Transpose[EigenA[s][[2]]]].VectorQ1[iota, s];
  Do[
   EigenVectorP[normal, j, s] = EigenA[s][[2, j]] * InverseProductP[normal, s][[j]];
   EigenVectorQ[normal, j, s] = EigenA[s][[2, j]] * InverseProductQ[normal, s][[j]];
   EigenVectorQ[iota, j, s] = EigenA[s][[2, j]] * InverseProductQ[iota, s][[j]];
   EigenValue[j, s] = EigenA[s][[1, j]];
   , {j, 1, 5}]
  , {s, {i, o}}]
```

Bound for k=0

Now we first implement the bound on the absolute values of the coefficients stated in Lemma 4.4.5,4.4.6 and Proposition 4.4.8

```
In[3786]:= Do[
  Bound[Xi, normal, 0, s] = Bound[ClosedRepBubble, 3, s]; (*Bound for Ξ,
  we extract the contribtion of Ξi_z^(0)(0)=1, explicitly in the analysis*)
  Bound[Xi, iota, 0, s] = Bound[ClosedRepBubble, 3, s] +
    rho[s] Bound[tG1, s] (1 + Bound[ClosedRepBubble, 3, s]);


  Bound[Xi, normal, 1, s] = rho[s] Bound[Q1, 0, s];
  Bound[Xi, iota, 1, s] = rho[s]/(2 d) Bound[Q1, iota, 0, s];
  factor[normal] = rho[s];
  factor[iota] = rho[s]/(2 d);
  Do[
   Bound[Xi, t, 2, s] = factor[t] VectorQ1[t, s].VectorAbar[s];
   Bound[Xi, t, 3, s] =
    factor[t] VectorQ1[t, s].MatrixAbar[s].VectorP1[normal, s];
   Bound[Xi, t, EvenTail, s] =
    Bound[Xi, t, 2, s] +
     Abs[
      factor[t]
       Sum[EigenVectorQ[t, j, s].MatrixAbar[s].VectorP1[normal, s] /
          (1 - EigenValue[j, s]^2) EigenValue[j, s], {j, 1, 5}]];
   Bound[Xi, t, OddTail, s] =
    Bound[Xi, t, 3, s] +
     Abs[
      factor[t]
       Sum[EigenVectorQ[t, j, s].MatrixAbar[s].VectorP1[normal, s] /
          (1 - EigenValue[j, s]^2) EigenValue[j, s]^2, {j, 1, 5}]];
   , {t, {normal, iota}}];
  Bound[Xi, normal, Even, s] = Bound[Xi, normal, EvenTail, s];
  (* recall here that extract the contribtion of Ξ^(0)(x)=δ_{0,x} in the analysis.*)
  Bound[Xi, iota, Even, s] = Bound[Xi, iota, 0, s] + Bound[Xi, iota, EvenTail, s];

  Do[
   Bound[Xi, t, Odd, s] = Bound[Xi, t, 1, s] + Bound[Xi, t, OddTail, s];
   Bound[Xi, t, Absolut, s] = Bound[Xi, t, Odd, s] + Bound[Xi, t, Even, s];
   , {t, {normal, iota}}];
  , {s, {i, o}}];
```

Bounds for $\hat{\Xi}^{(N)}(0)$-$\hat{\Xi}^{(N)}(k)$ for N=0,1,2,3

We now compute the bound as given in Lemma 4.4.5,4.4.6 and Proposition 4.4.8

In[3787]:=
```
Do[
  Bound[Xi, normal, 0, Delta, 0, s] = Bound[WeightedClosedBubble, 0, s];
  Bound[Xi, iota, 0, Delta, 0, s] =
   (1 + (2 d - 1) Bound[tG1i, s]) Bound[WeightedClosedBubble, 2, s];

  Bound[Xi, iota, 0, Delta, ei, s] =
   Bound[WeightedClosedBubble, 2, s] + 2 d Bound[tG1i, s] +
    2 (2 d Bound[tG1, s] Bound[ClosedRepBubble, 3, s] +
       2 d Bound[tG1, s] Bound[WeightedClosedBubble, 2, s]) ;

  Bound[Xi, iota, 1, Delta, 0, s] = rho[s] Bound[DeltaQ, I, 0, s];
  Bound[Xi, iota, 1, Delta, ei, s] = rho[s] Bound[DeltaQ, II, 0, s];
  Bound[Xi, normal, 1, Delta, 0, s] = rho[s] Bound[DeltaQ, start, 0, s];

  Bound[Xi, normal, 2, Delta, 0, s] =
   2 rho[s] (VectorQ1[normal, s].VectorDelta[end, s] +
      VectorDelta[startQ, s].VectorP1[normal, s]);
  Bound[Xi, iota, 2, Delta, 0, s] =
   2 rho[s]
    (VectorQ1[iota, s].VectorDelta[end, s] +
      VectorDelta[iotaIQ, s].VectorP1[normal, s]);
  Bound[Xi, iota, 2, Delta, ei, s] =
   2 rho[s]
    (VectorQ1[iota, s].VectorDelta[end, s] +
      VectorDelta[iotaIIQ, s].VectorP1[normal, s]);

  , {s, {i, o}}]
```

Bound for N=3

In[3788]:=
```
Do[
  Bound[Xi, normal, 3, Delta, 0, s] =
   3 rho[s] VectorDelta[startQ, s].MatrixA[s].VectorP1[normal, s] +
    3 rho[s] VectorQ1[normal, s].MatrixA[s].VectorDelta[end, s] +
    3 rho[s] VectorQ1[normal, s].MatrixDelta[s].VectorP1[normal, s];
  Bound[Xi, iota, 3, Delta, 0, s] =
   3 rho[s] VectorDelta[iotaIQ, s].MatrixA[s].VectorP1[normal, s] +
    3 rho[s] VectorQ1[iota, s].MatrixA[s].VectorDelta[end, s] +
    3 rho[s] VectorQ1[iota, s].MatrixDelta[s].VectorP1[normal, s];
  Bound[Xi, iota, 3, Delta, ei, s] =
   3 rho[s] VectorDelta[iotaIIQ, s].MatrixA[s].VectorP1[normal, s] +
    3 rho[s] VectorQ1[iota, s].MatrixA[s].VectorDelta[end, s] +
    3 rho[s] VectorQ1[iota, s].MatrixDelta[s].VectorP1[normal, s];
  , {s, {i, o}}]
```

Bounds for $\hat{\Xi}(0)$-$\hat{\Xi}(k)$ for N≥4

Bounds for $\hat{\Xi}(0)$-$\hat{\Xi}(k)$ for N>=4

We compute the sum over the bound of Proposition 4.4.8 over even N using the technique of Section 5.3.

In[3789]:=
```
Do[
   Bound[Xi, normal, EvenTail, Delta, 0, s] =
    Abs[
     rho[s] 2
      (Sum[VectorDelta[startQ, s].EigenVectorP[normal, j, s] EigenValue[j, s]^2
```

```
      (1/(1 - EigenValue[j, s]^2)^2 + 1/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
    Sum[EigenVectorQ[normal, j, s].VectorDelta[end, s] EigenValue[j, s]^2
      (1/(1 - EigenValue[j, s]^2)^2 + 1/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
    Sum[(EigenVectorQ[normal, j, s] EigenValue[j, s]^3) / (1 - EigenValue[j, s]^2)^2,
      {j, 1, 5}].Sum[EigenVectorP[normal, j, s]/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}] +
    Sum[(EigenVectorQ[normal, j, s] EigenValue[j, s]^2) / (1 - EigenValue[j, s]^2)^2,
      {j, 1, 5}].Sum[(EigenVectorP[normal, j, s] EigenValue[j, s]) /
       (1 - EigenValue[j, s]^2)^2, {j, 1, 5}])];
Bound[Xi, iota, EvenTail, Delta, 0, s] =
 Abs[
  rho[s] 2
   (Sum[VectorDelta[iotaIQ, s].EigenVectorP[normal, j, s] EigenValue[j, s]^2
      (1/(1 - EigenValue[j, s]^2)^2 + 1/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
    Sum[EigenVectorQ[iota, j, s].VectorDelta[end, s] EigenValue[j, s]^2
      (1/(1 - EigenValue[j, s]^2)^2 + 1/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
    Sum[(EigenVectorQ[iota, j, s] EigenValue[j, s]^3)/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
     Sum[EigenVectorP[normal, j, s]/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}] +
    Sum[(EigenVectorQ[iota, j, s] EigenValue[j, s]^2)/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
     Sum[(EigenVectorP[normal, j, s] EigenValue[j, s])/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}])];
Bound[Xi, iota, EvenTail, Delta, ei, s] =
 Abs[
  rho[s] 2
   (Sum[VectorDelta[iotaIIQ, s].EigenVectorP[normal, j, s] EigenValue[j, s]^2
      (1/(1 - EigenValue[j, s]^2)^2 + 1/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
    Sum[EigenVectorQ[iota, j, s].VectorDelta[end, s] EigenValue[j, s]^2
      (1/(1 - EigenValue[j, s]^2)^2 + 1/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
```

```
        Sum[(EigenVectorQ[iota, j, s] EigenValue[j, s]^3) / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
         Sum[EigenVectorP[normal, j, s] / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}] +
        Sum[(EigenVectorQ[iota, j, s] EigenValue[j, s]^2) / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
         Sum[(EigenVectorP[normal, j, s] EigenValue[j, s]) / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}])];
 , {s, {i, o}}];
```

Now we compute the sum over odd N

```
In[3790]:= Do[
  Bound[Xi, normal, OddTail, Delta, 0, s] =
   Abs[
    rho[s]
     (2 Sum[VectorDelta[startQ, s].EigenVectorP[normal, j, s] EigenValue[j, s]^2
          (1 / (1 - EigenValue[j, s]^2)^2 + 2 / (1 - EigenValue[j, s]^2)), {j, 1, 5}] +
       2 Sum[EigenVectorQ[normal, j, s].VectorDelta[end, s] EigenValue[j, s]^2
          (1 / (1 - EigenValue[j, s]^2)^2 + 2 / (1 - EigenValue[j, s]^2)), {j, 1, 5}] +
       2 Sum[(EigenVectorQ[normal, j, s] EigenValue[j, s]^3) / (1 - EigenValue[j, s]^2)^2,
          {j, 1, 5}].Sum[EigenVectorP[normal, j, s] / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}] -
       Sum[(EigenVectorQ[normal, j, s] EigenValue[j, s]^2) / (1 - EigenValue[j, s]^2)^2,
         {j, 1, 5}].Sum[(EigenVectorP[normal, j, s] EigenValue[j, s]) / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}])];
  Bound[Xi, iota, OddTail, Delta, 0, s] =
   Abs[
    rho[s]
     (2 Sum[VectorDelta[iotaIQ, s].EigenVectorP[normal, j, s] EigenValue[j, s]^2
          (1 / (1 - EigenValue[j, s]^2)^2 + 2 / (1 - EigenValue[j, s]^2)), {j, 1, 5}] +
       2 Sum[EigenVectorQ[iota, j, s].VectorDelta[end, s] EigenValue[j, s]^2
          (1 / (1 - EigenValue[j, s]^2)^2 + 2 / (1 - EigenValue[j, s]^2)), {j, 1, 5}] +
       2 Sum[(EigenVectorQ[iota, j, s] EigenValue[j, s]^3) / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
```

```
      Sum[EigenVectorP[normal, j, s] / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}] -
     Sum[EigenVectorQ[iota, j, s] EigenValue[j, s]^2 / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
      Sum[EigenVectorP[normal, j, s] EigenValue[j, s] / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}])];

Bound[Xi, iota, OddTail, Delta, ei, s] =
 Abs[
  rho[s]
   (2 Sum[VectorDelta[iotaIIQ, s].EigenVectorP[normal, j, s] EigenValue[j, s]^2
        (1 / (1 - EigenValue[j, s]^2)^2 + 2 / (1 - EigenValue[j, s]^2)), {j, 1, 5}] +
     2 Sum[EigenVectorQ[iota, j, s].VectorDelta[end, s] EigenValue[j, s]^2
        (1 / (1 - EigenValue[j, s]^2)^2 + 2 / (1 - EigenValue[j, s]^2)), {j, 1, 5}] +
     2 Sum[EigenVectorQ[iota, j, s] EigenValue[j, s]^3 / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
      Sum[EigenVectorP[normal, j, s] / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}] -
     Sum[EigenVectorQ[iota, j, s] EigenValue[j, s]^2 / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
      Sum[EigenVectorP[normal, j, s] EigenValue[j, s] / (1 - EigenValue[j, s]^2)^2, {j, 1, 5}])];
, {s, {i, o}}];
```

Summation of the Delta Bounds

In[3791]:=
```
Do[
 Do[
  Bound[Xi, t, Even, Delta, 0, s] =
   Bound[Xi, t, 0, Delta, 0, s] + Bound[Xi, t, 2, Delta, 0, s] +
    Bound[Xi, t, EvenTail, Delta, 0, s];
  Bound[Xi, t, Odd, Delta, 0, s] =
   Bound[Xi, t, 1, Delta, 0, s] + Bound[Xi, t, 3, Delta, 0, s] +
    Bound[Xi, t, EvenTail, Delta, 0, s];
  Bound[Xi, t, Absolut, Delta, 0, s] =
   Bound[Xi, t, Odd, Delta, 0, s] + Bound[Xi, t, Even, Delta, 0, s];
  , {t, {normal, iota}}];

 Bound[Xi, iota, Even, Delta, ei, s] =
  Bound[Xi, iota, 0, Delta, ei, s] + Bound[Xi, iota, 2, Delta, ei, s] +
   Bound[Xi, iota, EvenTail, Delta, ei, s];
 Bound[Xi, iota, Odd, Delta, ei, s] =
  Bound[Xi, iota, 1, Delta, ei, s] + Bound[Xi, iota, 3, Delta, ei, s] +
   Bound[Xi, iota, OddTail, Delta, ei, s];
 Bound[Xi, iota, Absolut, Delta, ei, s] =
  Bound[Xi, iota, Odd, Delta, ei, s] + Bound[Xi, iota, Even, Delta, ei, s];
 , {s, {i, o}}]
```

Computation of constants of Proposition 3.5.5

In this section we perform the computations of Section 3.4. to obtain the constances stated in Proposition 3.5.5:

$$\begin{gathered}\sum_x |F(x)| \leq K_F = \text{Bound[KF]} \\ \text{Bound[KPhi, 1]} = \underline{K}_\Phi \leq \hat{\Phi}(0) \leq \overline{K}_\Phi = \text{Bound[KPhi, 2]} \\ \text{Bound[KPhiabs, 1]} = \underline{K}_{|\Phi|} \leq \sum_x \left|\Phi(x)\right| \leq \overline{K}_{|\Phi|} = \text{Bound[KPhiabs, 2]} \\ \sum_{x\neq 0} |\Phi_z(x)| \leq K_{|\Phi|} = \text{Bound[KPhiWithoutZero]}\end{gathered} \tag{3}$$

$$\begin{gathered}\sum_x F(x)[1-\cos(k\,x)] \geq K_{\text{Lower}}\left[1-\hat{D}(k)\right] \\ \sum_x \left|F(x)\right|[1-\cos(k\,x)] \leq K_{\Delta\text{F}}\left[1-\hat{D}(k)\right] \\ \sum_x \left|\Phi_z(x)\right|[1-\cos(k\,x)] \leq K_{\Delta\Phi}\left[1-\hat{D}(k)\right]\end{gathered} \tag{4}$$

Bound on absolute value K_F and K_Φ

In[3792]:=
```
Do[
 alpha[s] = twodgjz[s] / (2 d);
 baralpha[s] = twodgz[s] / (2 d);
 Bound[KPsi, s] = rho[s] + (2 d - 2) / (2 d) Bound[Xi, normal, Absolut, s];
 Bound[KF, s] =
  (2 d baralpha[s]) / (1 - alpha[s] - (2 d - 2) alpha[s] Bound[Xi, iota, Absolut, s])
   Bound[KPsi, s];
 Bound[KPhiup, s] = 1 + Bound[Xi, normal, Even, s] +
   Bound[Xi, iota, Absolut, s] Bound[KF, s];
 Bound[KPhidown, s] = 1 - Bound[Xi, normal, Odd, s] -
   Bound[Xi, iota, Absolut, s] Bound[KF, s];
 Bound[KPhiabsup, s] = 1 + Bound[Xi, normal, Absolut, s] +
   Bound[Xi, iota, Absolut, s] Bound[KF, s];
 Bound[KPhiabsdown, s] = 1 - Bound[Xi, normal, Absolut, s] -
   Bound[Xi, iota, Absolut, s] Bound[KF, s];
 Bound[KPhiWithoutZero, s] =
  Bound[Xi, normal, Absolut, s] + Bound[Xi, iota, Absolut, s] Bound[KF, s];
 , {s, {i, o}}]
Bound[KPhiabsdown, o]
```

Out[3793]=
```
0.957113
```

Bounds on differences

Next we implement the computation of Section 3.4.3. First the differences of $F_1$and Φ_1, lines (3.4.26), (3.4.27), (3.4.29)

```
In[3794]:= Bound[DifferencefF, Part1, Lower, i] =
    2 d baralpha[i]
    ---------------
    1 - alpha[i]^2
    (1 - Bound[tG1i, i] - Bound[Xi, normal, Odd, Delta, 0, i] - Bound[Xi, normal, Odd, i] -
      alpha[i] Bound[Xi, normal, Even, Delta, 0, i]);
  Bound[DifferencefF, Part1, Absolut, i] =
    2 d baralpha[i]
    ---------------
    1 - alpha[i]^2
    (rho[i] + (1 + alpha[i]) Bound[Xi, normal, Absolut, Delta, 0, i] +
      Bound[Xi, normal, Absolut, i]);
  Bound[DifferencefF, Part1, Lower, o] =
    Min[2 d baralpha[i] / (1 - alpha[i]^2), 2 d baralpha[o] / (1 - alpha[o]^2)]
    (1 - Bound[tG1i, o] - Bound[Xi, normal, Odd, Delta, 0, o] - Bound[Xi, normal, Odd, o] -
      alpha[o] Bound[Xi, normal, Even, Delta, 0, o]);
  Bound[DifferencefF, Part1, Absolut, o] =
    Max[2 d baralpha[i] / (1 - alpha[i]^2), 2 d baralpha[o] / (1 - alpha[o]^2)]
    (rho[i] + (1 + alpha[o]) Bound[Xi, normal, Absolut, Delta, 0, o] +
      Bound[Xi, normal, Absolut, o]);
  Do[
   Bound[KDeltaPhi, Part1, s] = Bound[Xi, normal, Absolut, Delta, 0, s] +
      baralpha[s]
      --------------
      1 - alpha[s]^2
      (2 d Bound[Xi, normal, Absolut, Delta, 0, s] Bound[Xi, iota, Absolut, s] +
        (1 + Bound[Xi, normal, Absolut, s]) Bound[Xi, iota, Absolut, Delta, ei, s] +
        2 d alpha[s] Bound[Xi, normal, Absolut, Delta, 0, s]
         Bound[Xi, iota, Absolut, s] +
        alpha[s] (1 + Bound[Xi, normal, Absolut, s])
         Bound[Xi, iota, Absolut, Delta, 0, s]);
   , {s, {i, o}}]
```

Then the differences of F_2 and Φ_2: For the bound in (3.4.30), (3.4.32) and (3.4.35)

```
In[3799]:= Do[
  Bound[DifferencefF, Part2, Lower, s] =
   -((2 d baralpha[s])^2/(1 - alpha[s]^2))
     (Bound[Xi, normal, Odd, Delta, 0, s] Bound[Xi, iota, Odd, s] +
       Bound[Xi, normal, Even, Delta, 0, s] Bound[Xi, iota, Even, s])
    - (2 d baralpha[s]^2/(1 - alpha[s]^2)) (rho[s] + (2 d - 2)/(2 d) Bound[Xi, normal, Even, s])
     ( Bound[Xi, iota, Even, Delta, ei, s] + 2 d Bound[Xi, iota, Even, s] +
       alpha[s]^2 Bound[Xi, iota, Even, Delta, 0, s] +
       alpha[s] (Bound[Xi, iota, Odd, Delta, ei, s] + Bound[Xi, iota, Odd, Delta, 0, s] +
          2 d Bound[Xi, iota, Odd, s])) -
    (2 d baralpha[s]^2/(1 - alpha[s]^2)) ((2 d - 2)/(2 d) Bound[Xi, normal, Odd, s])
     ( Bound[Xi, iota, Odd, Delta, ei, s] + 2 d Bound[Xi, iota, Odd, s] +
       alpha[s]^2 Bound[Xi, iota, Odd, Delta, 0, s] +
       alpha[s] (Bound[Xi, iota, Even, Delta, ei, s] +
          Bound[Xi, iota, Even, Delta, 0, s] + 2 d Bound[Xi, iota, Even, s]));
  Bound[DifferencefF, Part2, Absolut, s] =
   ((2 d baralpha[s])^2/(1 - alpha[s]^2)) Bound[Xi, normal, Absolut, Delta, 0, s]
     Bound[Xi, iota, Absolut, s]
    + (2 d baralpha[s]^2/((1 - alpha[s]^2) (1 + alpha[s]))) (rho[s] + (2 d - 2)/(2 d) Bound[Xi, normal, Absolut, s])
     ( Bound[Xi, iota, Absolut, Delta, ei, s] + 2 d Bound[Xi, iota, Absolut, s] +
       alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s]);
  Bound[KDeltaPhi, Part2, s] =
   (2 d baralpha[s]^2/(1 - alpha[s]^2))
    (2 d Bound[Xi, normal, Absolut, Delta, 0, s] Bound[Xi, iota, Absolut, s]^2 +
     2 (1 + Bound[Xi, normal, Absolut, s])
      (Bound[Xi, iota, Absolut, s]/(1 + alpha[s]))
       (Bound[Xi, iota, Absolut, Delta, ei, s] +
         alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s]));
  , {s, {i, o}}]
```

Finally, we compute the differences of F_3 and Φ_3, lines (4.4.37) and (4.4.38)

In[3800]:=
```
Do[
 tmp = 1/(1 - (2 d alpha[s] Bound[Xi,iota,Absolut,s])/(1-alpha[s]));
 Bound[DifferencefF, Part3, Absolut, s] =
  Bound[Xi, normal, Absolut, Delta, 0, s] (2 d baralpha[s])/(1 - alpha[s])^3
    (2 d alpha[s] Bound[Xi, iota, Absolut, s])^2 tmp +
   (1 + Bound[Xi, normal, Absolut, s]) (baralpha[s] (2 d alpha[s])^2)/((1 - alpha[s])^2 (1 - alpha[s]^2))
    Bound[Xi, iota, Absolut, s] tmp^2
    (Bound[Xi, iota, Absolut, Delta, ei, s] +
      alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s]) +
   (1 + Bound[Xi, normal, Absolut, s]) (baralpha[s] (2 d alpha[s])^2)/((1 - alpha[s])^2 (1 - alpha[s]^2))
    Bound[Xi, iota, Absolut, s] tmp
    (Bound[Xi, iota, Absolut, Delta, ei, s] +
      alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s] +
      2 d Bound[Xi, iota, Absolut, s]);

 Bound[DifferencefF, Part3, Lower, s] = -Bound[DifferencefF, Part3, Absolut, s];
 Bound[KDeltaPhi, Part3, s] =
  Bound[Xi, normal, Absolut, Delta, 0, s] (2 d baralpha[s] Bound[Xi, iota, Absolut, s])/(1 - alpha[s])^3
    (2 d alpha[s] Bound[Xi, iota, Absolut, s])^2 tmp +
   (1 + Bound[Xi, normal, Absolut, s])
    (baralpha[s] (2 d alpha[s] Bound[Xi, iota, Absolut, s])^2) /
     ((1 - alpha[s])^2 (1 - alpha[s]^2)) (tmp^2 + tmp)
    1/(1 + alpha[s]) (Bound[Xi, iota, Absolut, Delta, ei, s] +
       alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s]);

 Bound[KDeltaFLower, s] =
  1 / (Bound[DifferencefF, Part1, Lower, s] + Bound[DifferencefF, Part2, Lower, s] +
     Bound[DifferencefF, Part3, Lower, s]);
 Bound[KDeltaF, s] = Bound[DifferencefF, Part1, Absolut, s] +
   Bound[DifferencefF, Part2, Absolut, s] + Bound[DifferencefF, Part3, Absolut, s];
 Bound[KDeltaPhi, s] = Bound[KDeltaPhi, Part1, s] + Bound[KDeltaPhi, Part2, s] +
   Bound[KDeltaPhi, Part3, s];
 Clear[tmp];
 , {s, {i, o}}]
```

Computation of additial bounds of Assumption 3.5.3

Now we compute bounds on α_F, α_Φ, $R_F$and R_Φ as given in Section 3.5.3 and 4.3.9. We follow the structure of Section 4.3.9. We know that $z \geq z_I = e^{-1}/(2d-1)$ and by simple combinatorics that

$$g_z^i \geq 1 + (2d-1)\, z_I (1 + (2d-1)\, z_I) + (2d-1)(2d-2)\frac{z_I^2}{2}$$

```
In[3801]:= z[i] = 1/((2 d - 1) Exp[1]);
gjzLower = 1 + (2 d - 1) z[i] (1 + (2 d - 1) z[i]) + ((2 d - 1) (2 d - 2))/2 z[i]^2;
gjjzLower = 1 + (2 d - 2) z[i] (1 + (2 d - 1) z[i]) + (2 d - 2)^2/2 z[i]^2;
gzLower = 1 + 2 d * z[i] + 2 × 2 d (2 d - 1) * z[i]^2 +
   2 d * (6 d - 4 + 6 d - 6 + (2 d - 2) / 2 (6 d - 8) + (2 d - 2) (6 d - 8)) z[i]^3 +
   (2 d (8 d - 6 + 8 d - 8) +
      2 d (2 d - 2) (1 / 2 + 8 d - 9 + 6 d - 6 + 4 d - 4 + 4 d - 4 + 4 d - 2 + 8 d - 12 + 6 d - 8) +
      2 d (2 d - 2) (2 d - 4) (2 d - 6) / 4! + 2 d (2 d - 2) (2 d - 4) / 2 * (2 d - 6)) z[i]^4;
z[o] = Gamma1/((2 d - 1) * gzLower);(* Upper bound on z and thereby also on z_c *)
alphaLower = z[i] gjzLower;
```

To rewrite Φ as in (3.5.31)-(3.5.33) we extract from $\Xi^{(0)}$ and $\Xi^{(1)}$ the nearest neighbor contribtion. For the implementation we split $\Xi^{(0)}$, $\Xi^{(1)}$ and $\Xi^{(0),i}$ as follows

```
In[3807]:= Bound[Xi, normalalphaPhi, 0, o] = Bound[ClosedRepLoop, 4, o];
Bound[Xi, normalRPhi, 0, o] = Bound[ClosedRepBubble, 4, o];

Bound[Xi, normalalphaPhi, 1, o] =
  rho[o] (2 Bound[ClosedRepLoop, 4, o] + Bound[ClosedRepBubble, 4, o]);
Bound[Xi, normalRPhi, 1, o] =
  rho[o] (Bound[ClosedRepLoop, 4, o] + 2 Bound[ClosedRepBubble, 4, o] +
      Bound[ClosedRepTriangle, 4, o]) +
   Sum[Bound[Q0, r, o] Bound[A, -r, 0, o], {r, 0, 2}];


Bound[Xi, iotaalphaPhi, 0, o] = rho[o] Bound[tG1i, o];
Bound[Xi, iotaRPhi, 0, o] =
  Bound[ClosedRepTriangle, 3, o] + rho[o] Bound[tG1, o] Bound[ClosedRepBubble, 3, o];
```

We use these quantities to define the bounds

```
In[3813]:= ap = Max[Bound[Xi, normalalphaPhi, 0, o],
   Bound[Xi, normalalphaPhi, 1, o] + twodgz[o]/(1 - gjz[o]^2) Bound[Xi, iotaalphaPhi, 0, o]];
Bound[Phi2Phi3, Absolut, 0, o] =
  (twodgz[o]/(1 - gjz[o]))^2 (d - 1)/d Bound[KPsi, o] Bound[Xi, iota, Absolut, o]^2
   1 / (1 - alpha[o] - (2 d - 2) alpha[o] Bound[Xi, iota, Absolut, o]);
 (*compare with (3.4.11)*)

bRp = Bound[Xi, normalRPhi, 0, o] + Bound[Xi, normalRPhi, 1, o] +
   Sum[Bound[Xi, normal, t, o], {t, {2, 3, EvenTail, OddTail}}] +
   twodgjz[o]/(1 - gjz[o]^2)
    (Bound[Xi, iotaRPhi, 0, o] + Sum[Bound[Xi, normal, t, o],
       {t, {1, 2, 3, EvenTail, OddTail}}]) +
   twodgz[o]/(1 - gjz[o]^2) Bound[Xi, normal, Absolut, o] Bound[Xi, iota, Absolut, o] +
   Bound[Phi2Phi3, Absolut, 0, o];
```

see (3.5.32)-(3.5.33).

To compute $\Phi(0) - \Phi(k)$ we compute the remainder term for the difference $\Phi_1(0) - \Phi_1(k)$:

```
In[3816]:= Bound[Xi, normalRPhi, 0, Delta, 0, o] = rho[o] Bound[WeightedClosedBubble, 2, o];
Bound[Xi, normalRPhi, 1, Delta, 0, o] =
  rho[o] (2 Bound[WeightedClosedBubble, 2, o] + Bound[WeightedClosedTriangle, 2, o] +
     2 (Sum[Bound[Q1, c, o] Bound[Delta, c, -2, o] +
          Bound[hQZero, c, o] Bound[A, c, 0, o], {c, 0, 2}]));

Bound[Xi, iotaRPhi, 0, DeltaPhi, 0, o] =
  twodgjz[o] Bound[tG3i, o] ((2 d - 1) Bound[tG1i, o]) +
   (2 d - 1) Bound[tG1i, o] rho[o] Bound[WeightedClosedBubble, 2, o];
Bound[Xi, iotaRPhi, 0, DeltaPhi, ei, o] =
  2 d Bound[Xi, iotaRPhi, 0, o] + Bound[Xi, iotaRPhi, 0, DeltaPhi, 0, o];


Bound[Xi, iotaRPhi, Absolut, o] =
  Bound[Xi, iota, Absolut, o] - Bound[Xi, iota, 0, o] + Bound[Xi, iotaRPhi, 0, o];
Bound[Xi, iotaRPhi, Absolut, Delta, 0, o] =
  Bound[Xi, iota, Absolut, Delta, 0, o] - Bound[Xi, iota, 0, Delta, 0, o] +
   Bound[Xi, iotaRPhi, 0, DeltaPhi, 0, o];
Bound[Xi, iotaRPhi, Absolut, Delta, ei, o] =
  Bound[Xi, iota, Absolut, Delta, ei, o] - Bound[Xi, iota, 0, Delta, ei, o] +
   Bound[Xi, iotaRPhi, 0, DeltaPhi, ei, o];
 (* TODO*)
```

Then, we use these differences and add the already computed differences $\Phi_2(0) - \Phi_2(k)$ and $\Phi_3(0) - \Phi_3(k)$:

```
In[3823]:= bRpDelta = Bound[Xi, normalRPhi, 0, Delta, 0, o] +
   Bound[Xi, normalRPhi, 1, Delta, 0, o] +
   Sum[Bound[Xi, normal, t, Delta, 0, o], {t, {2, 3, EvenTail, OddTail}}] +
   baralpha[o]
   -------------
   1 - alpha[o]^2
    (2 d Bound[Xi, normal, Absolut, Delta, 0, o] Bound[Xi, iotaRPhi, Absolut, o] +
      (1 + Bound[Xi, normal, Absolut, o]) Bound[Xi, iotaRPhi, Absolut, Delta, ei, o] +
      2 d alpha[o] Bound[Xi, normal, Absolut, Delta, 0, o]
       Bound[Xi, iotaRPhi, Absolut, o] +
      alpha[o] (1 + Bound[Xi, normal, Absolut, o])
       Bound[Xi, iotaRPhi, Absolut, Delta, 0, o]) +
   baralpha[o]
   -------------
   1 - alpha[o]^2
    (2 d alpha[o] Bound[Xi, normal, Absolut, Delta, 0, o]

      Bound[Xi, iotaalphaPhi, 0, o]
      -----------------------------  +
                 2 d

      alpha[o] Bound[Xi, normal, Absolut, o] Bound[Xi, iotaalphaPhi, 0, o]) +

   Bound[KDeltaPhi, Part2, o] + Bound[KDeltaPhi, Part3, o];
```

The first term corresponds are contributions to Ξ_z that have not been extracted. The second term bound $\sum_i \Psi^{(0),i}(0)\,(\hat{\Xi}^i(0) - \Xi^i(e_i)\big)$. The third term bounds $\sum_i \left(\hat{\Psi}^{(0),i}(0) - \Psi^{(0),i}(0)\right) \Xi^i(e_i)\big)$. In last term bound all remainder term the contribution of $\Phi_2$and Φ_3.

For the rewrite of $1 - F(k)$ we require the following quantities:

In[3824]:=
```
Psi1eiekLower = 6 z[i]^4 gjjzLower^4 + 20 (2 d - 2) z[i]^6 gjjzLower^6; (*Ψ^{1,k}(e1+e2)*)
Psi0e1Lower = (2 d - 2) z[i]^4 gjjzLower^4 + 4 (2 d - 2) (2 d - 4) z[i]^6 gjjzLower^6;
(*Ψ^{0,k}(e1)*)
Psi0e2Lower = (2 d - 3) z[i]^4 gjjzLower^4 + 4 (2 d - 3) (2 d - 4) z[i]^6 gjjzLower^6;
(*Ψ^{0,k}(e2)*)

Psi0eiek = gjz[o]^2 (gjz[o]^2 + c4ik/2 gjz[o]^4 + c6ik gjz[o]^6 + Bound[tG8ik, o]) +
   c4ik/2 gjz[o]^4 (c6ik gjz[o]^6 + Bound[tG8ik, o]) + Bound[tG6ik, o] Bound[tG6ik, o];
 (*Ψ^{0,k}(ei+ek)*)
Psi1e2 = Bound[ClosedRepBubble, 4, o]/twodgjz[o] Bound[tG1i, o] +
   Bound[ClosedRepBubble, 3, o]/twodgjz[o] Bound[OpenRepBubble, 2, o] Max[1, Bound[tG1, o]];
Psi1e1 = Psi1e2;
```

We use as bound on the absolute value of α_F(3.5.39) the following

In[3830]:=
```
af = 2 d gjz[o]/(1 - gjz[o]^2) + twodgz[o] (2 d - 2)/(1 - gjz[o]^2) (Psi0eiek - Psi1eiekLower) +
   twodgz[o] gjz[o]/(1 - gjz[o]^2) ((2 d - 2) (Psi1e2 - Psi0e2Lower) + (Psi1e1 - Psi0e1Lower));
```

For the computation of $F(0) - F(k)$ we use the following values

In[3831]:=
```
Bound[Xi, normalRf, 0, Delta, 0, o] = Bound[WeightedClosedBubble, 2, o];
Bound[Xi, normalRf, 1, Delta, 0, o] =
  2 Bound[WeightedClosedBubble, 2, o] + Bound[WeightedClosedTriangle, 2, o] +
   2 (Sum[Bound[Q1, c, o] Bound[Delta, c, -2, o] + Bound[hQZero, c, o] Bound[A, c, 0, o],
      {c, 0, 2}]);
Bound[Xi, normalRf, 0, o] = Bound[ClosedRepBubble, 4, o];
Bound[Xi, normalRf, 1, o] =
  (Bound[ClosedRepLoop, 4, o] + 2 Bound[ClosedRepBubble, 4, o] +
     Bound[ClosedRepTriangle, 4, o]) +
   2 (Sum[Bound[Q1, c, o] Bound[Delta, c, -2, o] + Bound[hQZero, c, o] Bound[A, c, 0, o],
      {c, 0, 2}]);
Bound[Xi, normalRf, Absolut, o] =
  Bound[Xi, normal, Absolut, o] +
   Sum[Bound[Xi, normalRf, t, o] - Bound[Xi, normal, t, o], {t, 0, 1}];
Bound[Xi, normalRf, Absolut, Delta, 0, o] =
  Bound[Xi, normal, Absolut, Delta, 0, o] +
   Sum[Bound[Xi, normalRf, t, Delta, 0, o] - Bound[Xi, normal, t, Delta, 0, o],
    {t, 0, 1}];
```

to create the bound s

In[3837]:=
$$\text{bRf} = \frac{\text{twodgz[o]}}{1-\text{gjz[o]}}\frac{2d-2}{2d}\text{Bound[Xi, normalRf, Absolut, o]} + \frac{\text{twodgz[o]}}{1-\text{gjz[o]}^2}\text{Bound[KPsi, o]}\frac{\frac{2d-2}{2d}\frac{\text{twodgz[o]}}{1-\text{gjz[o]}}\text{Bound[Xi, iota, Absolut, o]}}{1-\frac{\text{twodgz[o]}}{1-\text{gjz[o]}}\text{Bound[Xi, iota, Absolut, o]}};$$

$$\text{bRfDelta} = \frac{2\,d\,\text{baralpha[o]}}{1-\text{alpha[o]}^2}(\text{Bound[Xi, normalRf, Absolut, Delta, 0, o]} + \text{Bound[Xi, normalRf, Absolut, o]} + \text{alpha[o] Bound[Xi, normalRf, Absolut, Delta, 0, o]}) + \text{Bound[DifferencefF, Part2, Absolut, o]} + \text{Bound[DifferencefF, Part3, Absolut, o]};$$

Check of the sufficient condition

Now we can compute whether Q(γ,Γ,z) is satisfied, see Definition 3.5.6.

In[3839]:=
$$\text{Do}\Big[\ \text{NoBLEBoundF1[s]} = \frac{1+\frac{2d-2}{2d-1}\text{Gamma1 Bound[Xi, iota, Even, s]}}{\text{rho[s]} - \frac{2d-2}{2d}\text{Bound[Xi, normal, Odd, s]}};\quad \text{NoBLEBoundF2[s]} = \frac{2d-2}{2d-1}\text{Bound[KPhiabsup, s] Bound[KDeltaFLower, s]};\ , \{s, \{i, o\}\}\Big]$$

We finally check

In[3840]:=
```
Do[
 Succes[f1, s] = NoBLEBoundF1[s] < Gamma1;
 Succes[f2, s] = NoBLEBoundF2[s] < Gamma2;
 Succes[s] = Succes[f1, s] && Succes[f2, s];
 , {s, {i, o}}]
```

Further, we need the constants for the improvement of $\bar{f}_{3,n,l}$ and $\bar{f}_{4,n,l}$:

In[3841]:=
$$\text{BoundFFour[n\_, l\_]} := \text{BoundFFourBar}\Big[\text{n, l, }\frac{2d-2}{2d-1}\text{Gamma2, twodgz[s], 1, af, ap, bRf, bRp, bRfDelta, bRpDelta, Bound[KDeltaFLower, o]}\Big];$$

$$\text{BoundFThree[n\_, l\_]} := \text{BoundFThreeBar}\Big[\text{n, l, }\frac{2d-2}{2d-1}\text{Gamma2, twodgz[s], 1, af, ap, bRf, bRp, bRfDelta, bRpDelta, Bound[KDeltaFLower, o]}\Big];$$

In[3843]:=
```
Do[
 SuccesFThree[t + 1] = BoundFThree[1, t] < GammaThree[1, t];
 SuccesFThree[t + 5] = BoundFThree[2, t] < GammaThree[2, t];
 , {t, 0, 3}]
Succes[f3bar] = SuccesFThree[1] && SuccesFThree[2] && SuccesFThree[3] &&
   SuccesFThree[4] && SuccesFThree[5] && SuccesFThree[6] && SuccesFThree[7] &&
   SuccesFThree[8];

SuccesFFour[1] = BoundFFour[1, 4] < GammaFour[1, 4];
SuccesFFour[2] = BoundFFour[2, 4] < GammaFour[2, 4];
Succes[f4bar] = SuccesFFour[1] && SuccesFFour[2];
Succes[overall] = Succes[i] && Succes[o] && Succes[f3bar] && Succes[f4bar];
```

Result

The overall result

The statement that the bootstrap was succesful is

In[3849]:=
```
Succes[overall]
```

Out[3849]= True

If this succedes than the analysis of Section 3.3 can be used to proved mean-field behavoior for LA.

Assuming the bootstrap was succesful we prove the following bounds: The infrared bound in (3.3.12) and (3.3.13) hold with

In[3850]:=
```
(2 d - 2)/(2 d - 1) Gamma2(* ≥ G_z(k) [1-D̂(k)] *)
Max[Bound[KDeltaFLower, o], 1]
(* Nominator in (4.3.13) *)
```

Out[3850]= 1.14032

Out[3851]= 1.14878

Further, we have proven that $g_{z_c} z_c$ is smaller than

In[3852]:=
```
1/(2 d - 1) Gamma1
```

Out[3852]= 0.0256947

and that g_{z_c} smaller than

In[3853]:=
```
Gamma1 * Exp[1]
```

Out[3853]= 2.86367

The improvement of bounds

In[3854]:=
```
bubbles = {Graphics[{Green, Disk[{0, 0}, 0.8]}, ImageSize -> 30],
   Graphics[{Red, Disk[{0, 0}, 0.8]}, ImageSize -> 40]};
tableClassicCheck = {{Bounds, Init - f1, Init - f2, f1, f2, f̄4,1,4, f̄4,2,4},
   {Gamma, Gamma1, Gamma2, Gamma1, Gamma2 , GammaFour[1, 4], GammaFour[2, 4] },
   {Bounds, NoBLEBoundF1[i], NoBLEBoundF2[i], NoBLEBoundF1[o], NoBLEBoundF2[o],
    BoundFFour[1, 4], BoundFFour[2, 4]}, { check,
    If[Succes[f1, i], bubbles[[1]], bubbles[[2]]],
    If[Succes[f2, i], bubbles[[1]], bubbles[[2]]],
    If[Succes[f1, o], bubbles[[1]], bubbles[[2]]],
    If[Succes[f2, o], bubbles[[1]], bubbles[[2]]],
    If[SuccesFFour[1], bubbles[[1]], bubbles[[2]]],
    If[SuccesFFour[2], bubbles[[1]], bubbles[[2]]]}};
tableClassicCheckFthree =
  {{Bounds, "(1,0)", "(1,1)", "(1,2)", "(1,3)", "(2,0)", "(2,1)", "(2,2)", "(2,3)"},
   {Gamma, GammaThree[1, 0], GammaThree[1, 1], GammaThree[1, 2],
    GammaThree[1, 3] , GammaThree[2, 0], GammaThree[2, 1], GammaThree[2, 2],
    GammaThree[2, 3] }, {Bounds, BoundFThree[1, 0], BoundFThree[1, 1],
    BoundFThree[1, 2], BoundFThree[1, 3], BoundFThree[2, 0], BoundFThree[2, 1],
    BoundFThree[2, 2], BoundFThree[2, 3]}, { check,
    If[SuccesFThree[1], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[2], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[3], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[4], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[5], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[6], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[7], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[8], bubbles[[1]], bubbles[[2]]]}};
Labeled[Grid[tableClassicCheck, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {Automatic, Automatic, {{3, 3}, {2, 7}} → GrayLevel[0.7]}],
  Style["Result for f1 and f2 in  Dimension " Text[d], Bold], Top] // Text
Labeled[Grid[tableClassicCheckFthree, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True, 5 → True}},
   ItemStyle → {1 → Bold, 1 → Bold},
   Background → {Automatic, Automatic, {{3, 3}, {2, 9}} → GrayLevel[0.9]}],
  Style["Result for f3 in  Dimension " Text[d], Bold], Top] // Text
```

Result for f_1 and f_2 in Dimension 21

Out[3857]=

Bounds	Init – f_1	Init – f_2	f_1	f_2	$\bar{f}_{4,1,4}$	$\bar{f}_{4,2,4}$
Gamma	1.05348	1.16882	1.05348	1.16882	0.003472	0.00707836
Bounds	1.04894	1.08443	1.05348	1.16882	0.00347097	0.00707738
check						

Result for f_3 in Dimension 21

Out[3858]=

Bounds	(1,0)	(1,1)	(1,2)	(1,3)	(2,0)	(2,1)	(2,2)	(2,3)
Gamma	0.167251	0.0308847	0.012046	0.003472	0.248293	0.0545613	0.0215825	0.0070784
Bounds	0.167251	0.0308838	0.012045	0.00347097	0.248293	0.0545605	0.0215816	0.00707738
check								

In the following we implement a semi-automate proceedure to find appropiate value for the constants Γ_i. Initially we guess a good value for the constant and make a first computation. Then we deactivate there initial definition in the top of the document and use the code below.

We recompile the document a number of times and hope that the algorithm below converges to a fixed point.

In[3859]:=
```
(*{d,Gamma1,Gamma2,GammaThree[1,0],GammaThree[1,1],GammaThree[1,2],
  GammaThree[1,3],GammaThree[2,0],GammaThree[2,1],GammaThree[2,2],
  GammaThree[2,3],GammaFour[1,4],GammaFour[2,4]}
 Gamma1=NoBLEBoundF1[o]+0.000001;
Gamma2=NoBLEBoundF2[o]+0.000001;
Do[
  GammaThree[1,t]=BoundFThree[1,t]+0.000001;
  GammaThree[2,t]=BoundFThree[2,t]+0.000001;
  ,{t,0,3}]
 GammaFour[1,4]=BoundFFour[1,4]+0.000001;
GammaFour[2,4]=BoundFFour[2,4]+0.000001;
{d,Gamma1,Gamma2,GammaThree[1,0],GammaThree[1,1],GammaThree[1,2],
 GammaThree[1,3],GammaThree[2,0],GammaThree[2,1],GammaThree[2,2],
 GammaThree[2,3],GammaFour[1,4],GammaFour[2,4]}*)
```

Print out of the computed bounds in the coefficients

In[3860]:=
```
Do[
 MethodeFourTable[s] = {{Quantity, ΞZero, ΞOne, ΞTwo, ΞThree, ΞEvenTail, ΞOddTail},
    {Text[Bound for Ξ̂], Bound[Xi, normal, 0, s], Bound[Xi, normal, 1, s] ,
     Bound[Xi, normal, 2, s], Bound[Xi, normal, 3, s], Bound[Xi, normal, EvenTail, s],
     Bound[Xi, normal, OddTail, s]},
    {Text[Bound for Ξ̂ι], Bound[Xi, iota, 0, s], Bound[Xi, iota, 1, s] ,
     Bound[Xi, iota, 2, s], Bound[Xi, iota, 3, s], Bound[Xi, iota, EvenTail, s],
     Bound[Xi, iota, OddTail, s]},
    {Text[Ξ̂ (1 - cos (kx))], Bound[Xi, normal, 0, Delta, 0, s],
     Bound[Xi, normal, 1, Delta, 0, s], Bound[Xi, normal, 2, Delta, 0, s],
     Bound[Xi, normal, 3, Delta, 0, s], Bound[Xi, normal, EvenTail, Delta, 0, s],
     Bound[Xi, normal, OddTail, Delta, 0, s]},
    {Text[ Ξι (1 - cos (kx))], Bound[Xi, iota, 0, Delta, 0, s],
     Bound[Xi, iota, 1, Delta, 0, s], Bound[Xi, iota, 2, Delta, 0, s],
     Bound[Xi, iota, 3, Delta, 0, s], Bound[Xi, iota, EvenTail, Delta, 0, s],
     Bound[Xi, iota, OddTail, Delta, 0, s]},
    {Text[Ξι (1 - cos (k (x - eι)))] , Bound[Xi, iota, 0, Delta, ei, s],
     Bound[Xi, iota, 1, Delta, ei, s] , Bound[Xi, iota, 2, Delta, ei, s],
     Bound[Xi, iota, 3, Delta, ei, s], Bound[Xi, iota, EvenTail, Delta, ei, s],
     Bound[Xi, iota, OddTail, Delta, ei, s]}};
 , {s, {i, o}}]
MethodeFourTablePart1 = {{Quantity, KF, KPhi, DELTAFLower, DELTAFAbsolut, DELTAPhi},
   {Bound for , Bound[KF, o], Bound[KPhiup, o], Bound[KDeltaFLower, o],
    Bound[KDeltaF, o], Bound[KDeltaPhi, o]}};
MethodeFourTablePart2 =
  {{Quantity, DELTAFLower, 2, 3, DELTAFAbsolut, 2, 3, DELTAPhi, 2, 3},
   {Bound for , Bound[DifferencefF, Part1, Lower, o],
    Bound[DifferencefF, Part2, Lower, o], Bound[DifferencefF, Part3, Lower, o],
    Bound[DifferencefF, Part1, Absolut, o], Bound[DifferencefF, Part2, Absolut, o],
    Bound[DifferencefF, Part3, Absolut, o], Bound[KDeltaPhi, Part1, o],
    Bound[KDeltaPhi, Part2, o], Bound[KDeltaPhi, Part3, o]}};
MethodeFourTablePart3 = {{Quantity, αΦ, RΦ, ΔRΦ, αF, RF, ΔRF},
   {Bound for , ap, bRp, bRpDelta, af, bRf, bRfDelta}};


Labeled[Grid[MethodeFourTable[i], Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
```

```
  Background → {{None}, {GrayLevel[0.9]}, {None}}],
 Style["Bound on coefficients at zI in Dimension " Text[d], Bold], Top] // Text
Labeled[Grid[MethodeFourTable[o], Alignment → {Center}, Frame → True,
  Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
  Background → {{None}, {GrayLevel[0.9]}, {None}}],
 Style["Bound on coefficients in Dimension " Text[d], Bold], Top] // Text

Labeled[Grid[MethodeFourTablePart1, Alignment → {Center}, Frame → True,
  Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
  Background → {{None}, {GrayLevel[0.9]}, {None}}],
 Style["Bound on the constants of Proposition 3.5.5 in Dimension " Text[d], Bold],
 Top] // Text
Labeled[Grid[MethodeFourTablePart2, Alignment → {Center}, Frame → True,
  Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
  Background → {{None}, {GrayLevel[0.9]}, {None}}],
 Style["Bound on the constants of Proposition 3.5.5 in Dimension " Text[d], Bold],
 Top] // Text
Labeled[Grid[MethodeFourTablePart3, Alignment → {Center}, Frame → True,
  Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
  Background → {{None}, {GrayLevel[0.9]}, {None}}],
 Style["Element of the rewrite of the two-point function " Text[d], Bold], Top] //
Text
```

Bound on coefficients at z_I in Dimension 21

Out[3864]=

Quantity	$\Xi^{\mathbf{Zero}}$	$\Xi^{\mathbf{One}}$	$\Xi^{\mathbf{Two}}$	$\Xi^{\mathbf{Three}}$	$\Xi^{\mathbf{EvenTail}}$	$\Xi^{\mathbf{OddTail}}$
Bound for $\hat{\Xi}$	0.00155371	0.00594235	0.000325928	0.0000128844	0.000326202	0.0000128898
Bound for $\hat{\Xi}^{\iota}$	0.0262556	0.00129368	0.0000746925	1.94135×10^{-6}	0.0000747545	1.94262×10^{-6}
$(1-\cos kx)\hat{\Xi}$	0.00433853	0.0165355	0.00193613	0.000122315	1.63251×10^{-6}	2.44432×10^{-6}
$(1-\cos kx)\,\Xi^{\iota}$	0.00743625	0.0429584	0.0085538	0.000876244	8.96563×10^{-6}	0.0000134056
$\Xi^{\iota}(1-\cos k(x-e_{\iota}))$	1.0665	0.142204	0.0122989	0.000993283	0.00001155	0.0000172818

Bound on coefficients in Dimension 21

Out[3865]=

Quantity	$\Xi^{\mathbf{Zero}}$	$\Xi^{\mathbf{One}}$	$\Xi^{\mathbf{Two}}$	$\Xi^{\mathbf{Three}}$	$\Xi^{\mathbf{EvenTail}}$	$\Xi^{\mathbf{OddTail}}$
Bound for $\hat{\Xi}$	0.00202853	0.00801188	0.00067499	0.0000381557	0.000676212	0.0000381928
Bound for $\hat{\Xi}^{\iota}$	0.0281464	0.00177686	0.000149416	5.86811×10^{-6}	0.000149672	5.87613×10^{-6}
$(1-\cos kx)\hat{\Xi}$	0.00910318	0.0580511	0.0162763	0.00167809	0.0000328124	0.0000491776
$(1-\cos kx)\,\Xi^{\iota}$	0.0172279	0.13587	0.0628343	0.0114789	0.000162456	0.000243341
$\Xi^{\iota}(1-\cos k(x-e_{\iota}))$	1.1431	0.280065	0.0708542	0.0118564	0.000175091	0.000262291

Bound on the constants of Proposition 3.5.5 in Dimension 21

Out[3866]=

Quantity	**KF**	**KPhi**	**DELTAFLower**	**DELTAFAbsolut**	**DELTAPhi**
Bound for	1.13572	1.03484	1.14878	1.2492	0.129897

Bound on the constants of Proposition 3.5.5 in Dimension 21

Out[3867]=

Quantity	**DELTA⋱ FLow⋱ er**	**2**	**3**	**DELTA⋱ FAbs⋱ olut**	**2**	**3**	**DELTA⋱ Phi**	**2**	**3**
Bound for	0.941989	−0.0670⋱ 92	−0.0044⋱ 051⋱ 1	1.16726	0.07752⋱ 97	0.00440⋱ 511	0.127232	0.00257⋱ 22	0.00009⋱ 25829

Element of the rewrite of the two−point function 21

Out[3868]=

Quantity	α_Φ	R_Φ	ΔR_Φ	α_F	R_F	ΔR_F
Bound for	0.0309737	0.0237978	0.100062	1.07994	0.0596897	0.199569