SRW computation

Implementation of the computer-assisted proof of the paper "Generalized approach to the non-backtracking lace expansion" by Robert Fitzner and Remco van der Hofstad

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Abstract

This document is the first part of the computer-assisted proof of the non-backtracking lace expansion (NoBLE). The NoBLE was created by Remco van der Hofstad and the author to prove mean-field behavior for self-avoiding walk (SAW), lattice trees (LT), lattice animals (LA) and percolation. In this file the model-independent computations, explained in "Generalized approach to the non-backtracking lace expansion" Section 4, are computed. All references in this file are to this paper.

We compute the Green's function of simple random walk (SRW) and other related quantities. The computation of SRW-integrals are based upon techniques of Takashi Hara and Gordon Slade "The lace expansion for self-avoiding walk in five or more dimensions." (1991). Additionally, we store the number of self-avoiding and bond-avoiding walks that are used in the model-dependent files.

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What to do with this file

You use this file, together with the model-dependent file, to perform the computer-assisted part of the NoBLE. If successful, then we prove the infrared bound for a given model in a given dimension. The only input that is required is the dimension, see just below. Once selected you should compile the hole file and can then proceed with the model dependent file. You only need to return to this file if you want to change the dimensions that you consider.

This is the basic version of the SRW computations notebook. The NoBLE analysis makes use of the more involved version, called SRW.nb. We keep this file as it may be easier to read for inexperienced users.

Input

The dimension in which we perform the computations

```
d = 15;
param = 4; (*param=3, SAW if (d≥ 8), param=4 percolation (d≥ 15),
param=5 LT and LA and if d≥ 17*)
```

We compute the SRW-Bubbles, SRW-triangle, SRW-square and SRW-pentagon, and prescribe a precision of 20 digits. While the SRW-pentagon is finite for dimensions d≥11, *Mathematica* guarantees the prescibed precision only for d≥17. This is sufficient for LT and LA as the NoBLE works for these models only in d≥20. For percolation and SAW we do not require the SRW-pentagons. For param=4 the pentagon are not computed and the desired precision is garanteed by *Mathematica* for d≥15. For param=3 we also do not compute squares and can perform the computation for d≥8.

If you want to consider smaller dimensions or want to consider the techniques explain in the Appendix of the model-dependent papers, please use the advanced version of this file.

For the reader of this file

This file consists of three Sections:

- In Section 1, we compute the SRW integral $I_{n,l}$, which corresponds to the SRW Green's function.
- In Section 2, we compute bounds on other SRW-integrals, using the value of $I_{n,l}$.
- In Section 3, we store the number of self-avoiding and bond-avoiding walks.

In this basic version of the file the computations are done via build-in function of *Mathematica* or via explicit computation. This should also allows readers with only minor experience in programming to understand this file.

Next. we highlight a feature of the implementation. Namely, the coordinates which we use to identify points $x \in \mathbb{Z}^d$. We map the standard representation of a vector $x = (x_1, x_2, \ldots, x_d)$ as to

$$x \mid -- > v(x) = (N_1(x), N_2(x), N_3(x), N_4(x), \ldots.$$

where $N_j(x)$ = number of x_i with $|x| = j$. For example we represent x as follows

$$x = e_1 \mid -- > v(x) = \{1, 0, 0, 0, \ldots..\}$$
$$x = 4\, e_1 \mid -- > v(x) = \{0, 0, 0, 1, \ldots..\}$$
$$x = -4\, e_5 \mid -- > v(x) = \{0, 0, 0, 1, \ldots..\}$$
$$x = e_1 + e_2 + 2\, e_2 + 3_4 \mid -- > v(x) = \{2, 1, 1, 0, \ldots..\}$$

This mapping is not a bijection. However, it encodes all necessary informations as the considered SRW quantities have various symmetries, e.g. $I_{n,l}(2\, e_1 + e_2) = I_{n,l}(e_3 - 2\, e_5)$. Thus, this description via $v(x)$ carries all the information required to compute the value of a SRW-integral. We are only interested in x that are close to the origin, and we end our representation $v(x)$ after the last non-trivial entry, i.e. v(x)={0,1,1,0,.....}={0,1,1}.

1. Simple Random Walk integral $I_{n,l}(x)$

We compute the two-point function of the simple random walk,

$$I_{n,l}(x) = \int_{[-\pi,\pi]} e^{\mathrm{i}k\, x} \frac{\hat{D}^l(k)}{\left(1 - \hat{D}(k)\right)^n} \frac{d^d k}{(2\pi)^d}.$$

We use the following relations

$$I_{n,l}(x) = I_{n,(l-1)}(x) - I_{(n-1),(l-1)}(x).$$

to reduce the computation of $I_{n,l}(x)$ to the computation of $I_{0,l}(x)$ and $I_{n,0}(x)$. We split this section into four part:

- select the points x for which we want to compute $I_{n,l}(x)$
- compute $I_{0,l}(x)$
- compute $I_{n,0}(x)$
- use the relation above to compute $I_{n,l}(x)$

1.1 Selection of the x

To compute the bounds on the NoBLE coefficients, as explain the main part of the model dependent articles, we require the values of $I_{n,l}(x)$, $L_n(x)$ CITE for the following values of x:

```
NvecZero = {0};(* x=0 *)
Nveci = {1}; (* x= e1 *)
Nvecik = {2};(* x= e1+e2 *)
NvecTwoi = {0, 1};
(* x=2 e1 *)
```

To compute $L_n(x)$ for these values we also need to compute $I_{n,l}(x)$ for the following values (see also CITE) :

```
NvecThreei = {0, 0, 1}; (* x=3 e1 *)
NvecFouri = {0, 0, 0, 1}; (* x=4 e1 *)
NvecTwoiTwok = {0, 2}; (* x=2 e1+2 e2 *)
NvecTwoik = {1, 1}; (* x=2 e1+e2 *)
NvecThreeik = {1, 0, 1}; (* x=3 e1+e2 *)
NvecTwoikp = {2, 1}; (* x=2 e1+e2+e3 *)
Nvecikpj = {4};(* x= e1+e2+e3+e4 *)
NVecAll = {NvecZero, Nveci, NvecTwoi, NvecThreei, NvecFouri, Nvecik,
   NvecTwoikp, NvecTwoiTwok, NvecTwoik, NvecThreeik, Nvecikpj};
```

1.2 Computation of $I_{o,l}(x)$

Reviewing the defintiion it is easy to see that $(2\,d)^l\, I_{o,l}(x) = p_l(x)$, where $p_l(x)$ is the number of l –step simple random walk ending at x. We can compute the value of $p_l(x)$ using simple combinators, see Section 4.1.2. We did the combinatorics manually and save the data in the following:

```
MaxNumberOfSteps = 12;
Do[Do[NumberOfSRWTox[v, m] = 0, {m, 0, 12}], {v, NVecAll}]
(* p_n(0) *)
NumberOfSRWTox[{0}, 0] = 1;
NumberOfSRWTox[{0}, 2] = N[2 d];
NumberOfSRWTox[{0}, 4] = N[(d * 4!/(2 × 2) + (d (d - 1))/2 * 4!)];
NumberOfSRWTox[{0}, 6] = N[(d * 6!/(3! × 3!) + d * (d - 1) * 6!/(2 × 2) + (d (d - 1) (d - 2))/3! * 6!)];
NumberOfSRWTox[{0}, 8] =
  N[(d * 8!/(4! × 4!) + d * (d - 1) * (8!/(3! × 3!) + 8!/2^5) + (d (d - 1) (d - 2))/2 * 8!/(2 × 2) +
     (d (d - 1) (d - 2) (d - 3))/4! * 8!)];
NumberOfSRWTox[{0}, 10] =
  N[(d * 10!/(5! × 5!) + d * (d - 1) * (10!/(4! × 4!) + 10!/(3! × 3! × 2! × 2!)) +
     d (d - 1) (d - 2)/2 (10!/(3! × 3!) + 10!/(2!)^4) + (d (d - 1) (d - 2) (d - 3))/3! * 10!/(2! × 2!) +
     (d (d - 1) (d - 2) (d - 3) (d - 4))/5! 10!)];
NumberOfSRWTox[{0}, 12] =
  N[(d * 12!/(6! × 6!) + d * (d - 1) * 12! (1/(5! × 5!) + 1/(4! × 4! × 2! × 2!) + 1/((3!)^4 2!)) +
     d (d - 1) (d - 2) 12! (1/(4! × 4! × 2!) + 1/((3!)^2 (2!)^2) + 1/((2!)^6 3!)) +
     d (d - 1) (d - 2) (d - 3) 12! (1/(3! × 3! × 3!) + 1/((2!)^4 2! × 2!)) +
     d (d - 1) (d - 2) (d - 3) (d - 4) 12!/(2! × 2! × 4!) +
     12! 1/6! d (d - 1) (d - 2) (d - 3) (d - 4) (d - 5))];

(* p_n(e1) *)
```

$$\text{Do}\left[\text{NumberOfSRWTox}[\{1\}, m] = \frac{1}{2\,d}\ \text{NumberOfSRWTox}[\{0\}, m+1], \{m, \{1, 3, 5, 7, 9, 11\}\}\right]$$

(* p_n(2e_1) *)

$$\text{NumberOfSRWTox}[\{0, 1\}, 2] = 1;$$

$$\text{NumberOfSRWTox}[\{0, 1\}, 4] = \frac{4!}{3!} + (d-1)\ \frac{4!}{2!};$$

$$\text{NumberOfSRWTox}[\{0, 1\}, 6] = \frac{6!}{4! \times 2!} + (d-1)\ 6!\ \left(\frac{1}{(2!)^3} + \frac{1}{3!}\right) + \frac{(d-1)\ (d-2)}{2}\ \frac{6!}{2!};$$

$$\text{NumberOfSRWTox}[\{0, 1\}, 8] = \frac{8!}{5! \times 3!} + (d-1)\ 8!\ \left(\frac{1}{3! \times 3! \times 2!} + \frac{1}{2! \times 2! \times 3!} + \frac{1}{4! \times 2!}\right) + (d-1)\ (d-2)\ 8!\ \left(\frac{1}{(2!)^3} + \frac{1}{2 \times 3!}\right) + \frac{(d-1)\ (d-2)\ (d-3)}{3!}\ \frac{8!}{2!};$$

$$\text{NumberOfSRWTox}[\{0, 1\}, 10] = \frac{10!}{6! \times 4!} + (d-1)\ 10!\ \left(\frac{1}{4! \times 4! \times 2!} + \frac{1}{3! \times 3! \times 3!} + \frac{1}{2! \times 2! \times 4! \times 2!} + \frac{1}{5! \times 3!}\right) + (d-1)\ (d-2)\ 10!\ \left(\frac{1}{3! \times 3! \times 2!} + \frac{1}{2! \times 2!}\ \left(\frac{1}{(2!)^3\, 2} + \frac{1}{3!}\right) + \frac{1}{4! \times 2! \times 2}\right) + (d-1)\ (d-2)\ (d-3)\ 10!\ \left(\frac{1}{(2!)^3\, 2!} + \frac{1}{3! \times 3!}\right) + \frac{(d-1)\ (d-2)\ (d-3)\ (d-4)}{4!}\ \frac{10!}{2!};$$

$$\text{NumberOfSRWTox}[\{0, 1\}, 12] = \frac{12!}{7! \times 5!} + (d-1)\ 12!\ \left(\frac{1}{5! \times 5! \times 2!} + \frac{1}{4! \times 4! \times 3!} + \frac{1}{3! \times 3! \times 4! \times 2!} + \frac{1}{2! \times 2! \times 5! \times 3!} + \frac{1}{6! \times 4!}\right) + (d-1)\ (d-2)\ 12!\ \left(\frac{1}{4! \times 4! \times 2!} + \frac{1}{3! \times 3!}\ \left(\frac{1}{2! \times 2! \times 2!} + \frac{1}{3!}\right) + \frac{1}{2! \times 2!}\ \left(\frac{1}{(2!)^3\, 3!} + \frac{1}{4! \times 2!}\right) + \frac{1}{2!}\ \frac{1}{5! \times 3!}\right) + (d-1)\ (d-2)\ (d-3)\ 12!\ \left(\frac{1}{3! \times 3! \times 2! \times 2!} + \frac{1}{2! \times 2!}\ \left(\frac{1}{(2!)^3\, 2!} + \frac{1}{3! \times 2!}\right) + \frac{1}{3!}\ \frac{1}{4! \times 2!}\right) + (d-1)\ (d-2)\ (d-3)\ (d-4)\ 12!\ \left(\frac{1}{2! \times 2! \times 3! \times 2!} + \frac{1}{4! \times 3!}\right) + \frac{1}{5!}\ (d-1)\ (d-2)\ (d-3)\ (d-4)\ (d-5)\ \frac{12!}{2!};$$

(* p_n(3e_1) *)

$$\text{NumberOfSRWTox}[\{0, 0, 1\}, 3] = 1;$$

$$\text{NumberOfSRWTox}[\{0, 0, 1\}, 5] = \frac{5!}{4!} + (d-1)\ \frac{5!}{3!};$$

$$\text{NumberOfSRWTox}[\{0, 0, 1\}, 7] = \frac{7!}{5! \times 2!} + (d-1)\ \left(\frac{7!}{2! \times 2! \times 3!} + \frac{7!}{4!}\right) + (d-1)\ (d-2)\ \frac{7!}{3! \times 2!};$$

$$\text{NumberOfSRWTox}[\{0, 0, 1\}, 9] = \frac{9!}{6! \times 3!} + (d-1)\ \left(\frac{9!}{3! \times 3! \times 3!} + \frac{9!}{2! \times 2! \times 4! \times 1!} + \frac{9!}{1! \times 1! \times 5! \times 2!}\right) + (d-1)\ (d-2)\ \left(\frac{9!}{2! \times 2! \times 3!} + \frac{9!}{4! \times 2!}\right) + (d-1)\ (d-2)\ \frac{(d-3)\ 9!}{3! \times 3!};$$

$$\text{NumberOfSRWTox}[\{0, 0, 1\}, 11] =$$

$$\frac{11!}{7!\times 4!}+$$
$$(d-1)\ 11!\left(\frac{1}{4!\times 4!\times 3!}+\frac{1}{3!\times 3!\times 4!\times 1!}+\frac{1}{2!\times 2!\times 5!\times 2!}+\frac{1}{1!\times 1!\times 6!\times 3!}\right)+$$
$$(d-1)\ (d-2)\ 11!\left(\frac{1}{3!\times 3!\times 3!}+\frac{1}{2!\times 2!}\left(\frac{1}{2!\times 2!\times 2!\times 3!}+\frac{1}{4!}\right)+\frac{1}{2!\times 5!\times 2!}\right)+$$
$$(d-1)\ (d-2)\ (d-3)\ 11!\left(\frac{1}{2!\times 2!\times 2!\times 3!}+\frac{1}{3!\times 4!}\right)+$$
$$(d-1)\ (d-2)\ (d-3)\ \frac{(d-4)\ 11!}{4!\times 3!};$$

(* p_n(4e₁) *)

$$\text{NumberOfSRWTox}[\{0,0,0,1\},4]=1;$$

$$\text{NumberOfSRWTox}[\{0,0,0,1\},6]=\frac{6!}{5!\times 1!}+(d-1)\ \frac{6!}{4!};$$

$$\text{NumberOfSRWTox}[\{0,0,0,1\},8]=$$
$$\frac{8!}{6!\times 2!}+(d-1)\ 8!\left(\frac{1}{(2!)^2\,4!}+\frac{1}{5!}\right)+\frac{(d-1)\ (d-2)}{2}\ \frac{8!}{4!};$$

$$\text{NumberOfSRWTox}[\{0,0,0,1\},10]=$$
$$\frac{10!}{7!\times 3!}+(d-1)\ 10!\left(\frac{1}{3!\times 3!\times 4!}+\frac{1}{2!\times 2!\times 5!}+\frac{1}{6!\times 2!}\right)+$$
$$(d-1)\ (d-2)\ 10!\left(\frac{1}{(2!)^2\,4!}+\frac{1}{2\times 5!}\right)+\frac{(d-1)\ (d-2)\ (d-3)}{3!}\ \frac{10!}{4!};$$

$$\text{NumberOfSRWTox}[\{0,0,0,1\},12]=$$
$$\frac{12!}{8!\times 4!}+(d-1)\ 12!\left(\frac{1}{4!\times 4!\times 4!}+\frac{1}{3!\times 3!\times 5!}+\frac{1}{2!\times 2!\times 6!\times 2!}+\frac{1}{7!\times 3!}\right)+$$
$$(d-1)\ (d-2)\ 12!\left(\frac{1}{(3!)^2\,4!}+\frac{1}{(2!)^5\,4!}+\frac{1}{2!\times 2!\times 5!}+\frac{1}{2\times 6!\times 2!}\right)+$$
$$(d-1)\ (d-2)\ (d-3)\ 12!\left(\frac{1}{2!\times 2!\times 2!\times 4!}+\frac{1}{3!\times 5!}\right)+$$
$$\frac{(d-1)\ (d-2)\ (d-3)\ (d-4)}{4!}\ \frac{12!}{4!};$$

(* p_n(e₁+e₂) *)

$$\text{NumberOfSRWTox}[\{2\},2]=2;$$

$$\text{NumberOfSRWTox}[\{2\},4]=2*\frac{4!}{2!}+(d-2)\ 4!;$$

$$\text{NumberOfSRWTox}[\{2\},6]=6!\left(\frac{2}{3!\times 2!}+\frac{1}{(2!)^2}\right)+(d-2)\ 6!\left(\frac{1}{2!\times 2!}+\frac{2}{2!}\right)+$$
$$(d-2)\ \frac{(d-3)}{2}\ 6!;$$

$$\text{NumberOfSRWTox}[\{2\},8]=2*8!\left(\frac{1}{4!\times 3!}+\frac{1}{3!\times 2!\times 2!\times 1!}\right)+$$
$$(d-2)\ 8!\left(\frac{1}{3!\times 3!}+\frac{2}{2!\times 2!\times 2!\times 1!\times 1!}+\frac{1}{(2!\times 1!)^2}+\frac{2}{3!\times 2!}\right)+$$
$$(d-2)\ (d-3)\ 8!\left(\frac{1}{2!\times 2!}+\frac{2}{2!\times 2!}\right)+\frac{(d-2)\ (d-3)\ (d-4)}{3!}\ 8!;$$

$$\text{NumberOfSRWTox}[\{2\},10]=2*10!\left(\frac{1}{5!\times 4!}+\frac{1}{4!\times 3!\times 2!}+\frac{1}{(3!\times 2!)^2\,2!}\right)+$$

$$(d-2)\,10!\left(\frac{1}{4!\times4!}+\frac{2}{3!\times3!\times2!\times1!\times1!}+\frac{1}{2!\times2!\times2!\times2!}+\frac{2}{2!\times2!\times3!\times2!}+\frac{2}{4!\times3!}+\frac{2}{3!\times2!\times2!}\right)+$$
$$(d-2)(d-3)\,10!\left(\frac{1}{3!\times3!}+\frac{1}{(2!)^5}+\frac{2}{2!\times2!\times2!}+\frac{2}{2!\times3!\times2!}+\frac{1}{2!\times2!\times2!}\right)+$$
$$(d-2)(d-3)(d-4)\,10!\left(\frac{1}{2!\times2!\times2!}+\frac{2}{3!\times2!}\right)+(d-2)(d-3)(d-4)(d-5)\,\frac{10!}{4!};$$

$$\text{NumberOfSRWTox}[\{2\},12]=2*12!\left(\frac{1}{6!\times5!}+\frac{1}{5!\times4!\times2!}+\frac{1}{4!\times3!\times3!\times2!}\right)+$$
$$(d-2)\,12!\left(\frac{1}{5!\times5!}+\frac{2}{4!\times4!\times2!}+\frac{2}{3!\times3!\times3!\times2!}+\frac{1}{3!\times3!\times2!\times2!}+\frac{2}{2!\times2!\times4!\times3!}+\frac{2}{2!\times2!\times2!\times3!\times2!}+\frac{2}{5!\times4!}+\frac{2}{4!\times3!\times2!}+\frac{1}{3!\times2!\times3!\times2!}\right)+$$
$$(d-2)(d-3)\,12!\left(\frac{1}{4!\times4!}+\frac{1}{3!\times3!\times2!\times2!}+\frac{2}{3!\times3!\times2!}+\frac{2}{(2!)^5\,2!}+\frac{1}{(2!)^4}+\frac{2}{2!\times2!\times3!\times2!}+\frac{2}{4!\times3!\times2!}+\frac{2}{3!\times2!\times2!\times2!}\right)+$$
$$(d-2)(d-3)(d-4)\,12!\left(\frac{1}{3!\times3!\times2!}+\frac{1}{(2!)^5}+\frac{2}{(2!)^4}+\frac{2}{3!\times2!\times3!}+\frac{1}{3!\times2!\times2!}\right)+$$
$$(d-2)(d-3)(d-4)(d-5)\,12!\left(\frac{1}{2!\times2!\times3!}+\frac{2}{4!\times2!}\right)+$$
$$(d-2)(d-3)(d-4)(d-5)(d-6)\,\frac{12!}{5!};$$

(* p_n($2e_1+2e_2$) *)

$$\text{NumberOfSRWTox}[\{0,2\},4]=\frac{4!}{2!\times2!};$$

$$\text{NumberOfSRWTox}[\{0,2\},6]=2\,\frac{6!}{2!\times3!}+(d-2)\,\frac{6!}{2!\times2!};$$

$$\text{NumberOfSRWTox}[\{0,2\},8]=\left(\frac{8!}{3!\times3!}+\frac{2\times8!}{4!\times2!\times2!}\right)+(d-2)\,8!\left(\frac{1}{2!\times2!\times2!\times2!}+\frac{2}{3!\times2!}\right)+\frac{(d-2)(d-3)}{2}\,\frac{8!}{2!\times2!};$$

$$\text{NumberOfSRWTox}[\{0,2\},10]=10!\left(\frac{2}{5!\times3!\times2!}+\frac{2}{4!\times2!\times3!\times1!}\right)+$$
$$(d-2)\,10!\left(\frac{2}{4!\times2!\times2!}+\frac{2}{2!\times2!\times3!\times2!}+\frac{1}{3!\times3!}\right)+$$
$$(d-2)(d-3)\,10!\left(\frac{1}{(2!)^4}+\frac{2}{3!\times2!\times2}\right)+(d-2)(d-3)\,\frac{(d-4)}{3!}\,\frac{10!}{2!\times2!};$$

$$\text{NumberOfSRWTox}[\{0,2\},12]=12!\left(\frac{2}{5!\times5!\times2!}+\frac{2}{4!\times4!\times3!}+\frac{1}{(4!\times2!)\times2!}\right)+$$
$$(d-2)\,12!\left(\frac{1}{4!\times4!\times2!\times2!}+\frac{2}{3!\times3!\times3!\times2!}+\frac{2}{2!\times2!\times4!\times2!}+\frac{1}{2!\times2!\times3!\times3!}+\frac{2}{5!\times3!\times2!}+\frac{2}{4!\times2!\times3!}\right)+$$
$$(d-2)(d-3)\,12!$$

$$\left(\frac{1}{2!\times2!\times3!\times2!}+\frac{1}{(2!)^{5}\,2!\times2!}+\frac{1}{2!\times2!\times3!\times2!}+\frac{2}{4!\times2!\times2!}+\frac{1}{3!\times3!}\right)+$$
$$(d-2)\,(d-3)\,(d-4)\,12!\left(\frac{2}{3!\times2!\times3!}+\frac{1}{(2!)^{5}}\right)+(d-2)\,(d-3)\,(d-4)\,\frac{(d-5)}{4!}\,\frac{12!}{2!\times2!};$$

(* p_n(2e$_1$+e$_2$) *)

$$\text{NumberOfSRWTox}[\{1,1\},3]=\frac{3!}{2!};$$

$$\text{NumberOfSRWTox}[\{1,1\},5]=\frac{5!}{3!}+\frac{5!}{2!\times2!}+(d-2)\,\frac{5!}{2!};$$

$$\text{NumberOfSRWTox}[\{1,1\},7]=$$
$$7!\left(\frac{1}{4!\times2!}+\frac{1}{3!\times2!}+\frac{1}{2!\times3!\times2!}\right)+(d-2)\,7!\left(\frac{1}{2!\times2!\times2!}+\frac{1}{3!}+\frac{1}{2!\times2!}\right)+$$
$$\frac{(d-2)\,(d-3)}{2}\,\frac{7!}{2!};$$

$$\text{NumberOfSRWTox}[\{1,1\},9]=9!\left(\frac{1}{5!\times3!}+\frac{1}{4!\times2!\times2!}+\frac{1}{3!\times3!\times2!}+\frac{1}{2!\times4!\times3!}\right)+$$
$$(d-2)\,9!\left(\frac{1}{3!\times3!\times2!}+\frac{1}{2!\times2!\times3!}+\frac{1}{2!\times2!\times2!\times2!}+\frac{1}{4!\times2!}+\frac{1}{3!\times2!}+\right.$$
$$\left.\frac{1}{2!\times3!\times2!}\right)+(d-2)\,(d-3)\,9!\left(\frac{1}{2!\times2!\times2!}+\frac{1}{3!\times2!}+\frac{1}{2!\times2!\times2!}\right)+$$
$$\frac{(d-2)\,(d-3)\,(d-4)}{3!}\,\frac{9!}{2!};$$

$$\text{NumberOfSRWTox}[\{1,1\},11]=$$
$$11!\left(\frac{1}{6!\times4!}+\frac{1}{5!\times3!\times2!}+\frac{1}{4!\times2!\times3!\times2!}+\frac{1}{3!\times4!\times3!}+\frac{1}{2!\times5!\times4!}\right)+$$
$$(d-2)\,11!$$
$$\left(\frac{1}{4!\times4!\times2!}+\frac{1}{3!\times3!}\left(\frac{1}{3!}+\frac{1}{2!\times2!}\right)+\frac{1}{2!\times2!}\left(\frac{1}{4!\times2!}+\frac{1}{3!\times2!}+\frac{1}{2!\times3!\times2!}\right)+\right.$$
$$\left.\frac{1}{5!\times3!}+\frac{1}{4!\times2!\times2!}+\frac{1}{3!\times3!\times2!}+\frac{1}{2!\times4!\times3!}\right)+$$
$$(d-2)\,(d-3)\,11!$$
$$\left(\frac{1}{3!\times3!\times2!}+\frac{1}{(2!)^{5}\,2!}+\frac{1}{2!\times2!\times3!}+\frac{1}{2!\times2!\times2!\times2!}+\frac{1}{2!\times4!\times2!}+\right.$$
$$\left.\frac{1}{2!\times3!\times2!}+\frac{1}{2!\times2!\times3!\times2!}\right)+$$
$$(d-2)\,(d-3)\,(d-4)\,11!\left(\frac{1}{(2!)^{4}}+\frac{1}{3!\times3!}+\frac{1}{3!\times2!\times2!}\right)+$$
$$\frac{(d-2)\,(d-3)\,(d-4)\,(d-5)}{4!}\,\frac{11!}{2!};$$

(* p_n(3e$_1$+e$_2$) *)

$$\text{NumberOfSRWTox}[\{1,0,1\},4]=\frac{4!}{3!};$$

$$\text{NumberOfSRWTox}[\{1,0,1\},6]=6!\left(\frac{1}{4!}+\frac{1}{3!\times2!}\right)+(d-2)\,\frac{6!}{3!};$$

$$\text{NumberOfSRWTox}[\{1,0,1\},8]=$$
$$8!\left(\frac{1}{5!\times2!}+\frac{1}{4!\times1!\times2!}+\frac{1}{3!\times3!\times2!}\right)+(d-2)\,8!\left(\frac{1}{2!\times2!\times3!}+\frac{1}{4!}+\frac{1}{3!\times2!}\right)+$$

$$\frac{(d-2)\,(d-3)}{2}\,\frac{8\,!}{3\,!};$$

$$\text{NumberOfSRWTox}[\{1,0,1\},10] = 10\,!\left(\frac{1}{6\,!\times 3\,!}+\frac{1}{5\,!\times 2\,!\times 2\,!}+\frac{1}{4\,!\times 3\,!\times 2\,!}+\frac{1}{3\,!\times 4\,!\times 3\,!}\right)+ (d-2)\,10\,!\left(\frac{1}{3\,!\times 3\,!\times 3\,!}+\frac{1}{2\,!\times 2\,!\times 4\,!}+\frac{1}{2\,!\times 2\,!\times 3\,!\times 2\,!}+\frac{1}{5\,!\times 2\,!}+\frac{1}{4\,!\times 2\,!}+\frac{1}{3\,!\times 3\,!\times 2\,!}\right)+(d-2)\,(d-3)\,10\,!\left(\frac{1}{2\,!\times 2\,!\times 3\,!}+\frac{1}{4\,!\times 2\,!}+\frac{1}{3\,!\times 2\,!\times 2\,!}\right)+ (d-2)\,(d-3)\,\frac{(d-4)}{3\,!}\,\frac{10\,!}{3\,!};$$

$$\text{NumberOfSRWTox}[\{1,0,1\},12] = 12\,!\left(\frac{1}{7\,!\times 4\,!}+\frac{1}{6\,!\times 3\,!\times 2\,!}+\frac{1}{5\,!\times 2\,!\times 3\,!\times 2\,!}+\frac{1}{4\,!\times 1\,!\times 4\,!\times 3\,!}+\frac{1}{3\,!\times 5\,!\times 4\,!}\right)+ (d-2)\,12\,!\left(\frac{1}{4\,!\times 4\,!\times 3\,!}+\frac{1}{3\,!\times 3\,!}\left(\frac{1}{4\,!}+\frac{1}{3\,!\times 2\,!}\right)+\frac{1}{2\,!\times 2\,!}\left(\frac{1}{5\,!\times 2\,!}+\frac{1}{4\,!\times 2\,!}+\frac{1}{3\,!\times 3\,!\times 2\,!}\right)+ \frac{1}{6\,!\times 3\,!}+\frac{1}{5\,!\times 2\,!\times 2\,!}+\frac{1}{4\,!\times 3\,!\times 2\,!}+\frac{1}{3\,!\times 4\,!\times 3\,!}\right)+ (d-2)\,(d-3)\,12\,!\left(\frac{1}{3\,!\times 3\,!\times 3\,!}+\frac{1}{(2\,!)^5\,3\,!}+\frac{1}{2\,!\times 2\,!}\left(\frac{1}{4\,!}+\frac{1}{3\,!\times 2\,!}\right)+ \frac{1}{2\,!}\left(\frac{1}{5\,!\times 2\,!}+\frac{1}{4\,!\times 2\,!}+\frac{1}{3\,!\times 3\,!\times 2\,!}\right)\right)+ (d-2)\,(d-3)\,(d-4)\,12\,!\left(\frac{1}{(2\,!)^3\,3\,!}+\frac{1}{3\,!\times 4\,!}+\frac{1}{3\,!\times 3\,!\times 2\,!}\right)+ \frac{(d-2)\,(d-3)\,(d-4)\,(d-5)}{4\,!}\,\frac{12\,!}{3\,!};$$

(* p_n($2e_1+e_2+e_3$) *)

$$\text{NumberOfSRWTox}[\{2,1\},4] = \frac{4\,!}{2\,!};$$

$$\text{NumberOfSRWTox}[\{2,1\},6] = 6\,!\left(\frac{2}{2\,!\times 2\,!}+\frac{1}{3\,!}\right)+(d-3)\,\frac{6\,!}{2\,!};$$

$$\text{NumberOfSRWTox}[\{2,1\},8] = 8\,!\left(\frac{2}{3\,!\times 2\,!\times 2\,!}+\frac{1}{2\,!\times 2\,!\times 2\,!}+\frac{2}{2\,!\times 3\,!}+\frac{1}{4\,!\times 2\,!}\right)+ (d-3)\,8\,!\left(\frac{2}{2\,!\times 2\,!}+\frac{1}{3\,!}+\frac{1}{2\,!\times 2\,!\times 2\,!}\right)+\frac{(d-3)\,(d-4)}{2}\,\frac{8\,!}{2\,!};$$

$$\text{NumberOfSRWTox}[\{2,1\},10] = 10\,!\left(\frac{2}{4\,!\times 3\,!\times 2\,!}+\frac{2}{3\,!\times 2\,!\times 2\,!\times 2\,!}+\frac{2}{3\,!\times 3\,!\times 2\,!}+\frac{1}{2\,!\times 2\,!\times 3\,!}+\frac{2}{2\,!\times 4\,!\times 2\,!}+\frac{1}{5\,!\times 3\,!}\right)+ (d-3)\,10\,!\left(\frac{2}{3\,!\times 2\,!\times 2\,!}+\frac{1}{2\,!\times 2\,!\times 2\,!}+\frac{2}{2\,!\times 3\,!}+\frac{2}{2\,!\times 2\,!\times 2\,!\times 2\,!}+\frac{1}{4\,!\times 2\,!}+ \frac{1}{3\,!\times 2\,!\times 2\,!}+\frac{1}{2\,!\times 3\,!\times 3\,!}\right)+ (d-3)\,(d-4)\,10\,!\left(\frac{2}{2\,!\times 2\,!\times 2\,!}+\frac{1}{3\,!\times 2\,!}+\frac{1}{2\,!\times 2\,!\times 2\,!}\right)+\frac{(d-3)\,(d-4)\,(d-5)}{3\,!}\,\frac{10\,!}{2\,!};$$

$$
\begin{aligned}
&\text{NumberOfSRWTox}[\{2,1\},12] = \\
&\quad 12!\left(\frac{1}{6!\times 4!}+\frac{2}{5!\times 3!\times 2!}+\frac{2}{4!\times 2!\times 3!\times 2!}+\frac{1}{4!\times 2!\times 2!\times 2!}+\frac{2}{3!\times 4!\times 3!}+\right.\\
&\qquad\left.\frac{2}{3!\times 3!\times 2!\times 2!}+\frac{2}{2!\times 5!\times 4!}+\frac{2}{2!\times 4!\times 3!\times 2!}+\frac{1}{2!\,(3!\times 2!)^2}\right)+\\
&\quad (d-3)\,12!\left(\frac{1}{4!\times 4!\times 2!}+\frac{1}{3!\times 3!\times 3!}+\frac{2}{3!\times 3!\times 2!\times 2!}+\frac{1}{2!\times 2!\times 4!\times 2!}+\right.\\
&\qquad \frac{2}{2!\times 2!\times 3!\times 2!}+\frac{2}{(2!)^3\,3!\times 2!}+\frac{1}{(2!)^5}+\frac{1}{5!\times 3!}+\frac{2}{4!\times 2!\times 2!}+\\
&\qquad\left.\frac{2}{3!\times 3!\times 2!}+\frac{1}{3!\times 2!\times 2!}+\frac{2}{2!\times 4!\times 3!}+\frac{2}{2!\times 3!\times 2!\times 2!}\right)+\\
&\quad (d-3)\,(d-4)\,12!\\
&\qquad\left(\frac{1}{3!\times 3!\times 2!}+\frac{1}{2^6}+\frac{1}{2!\times 2!\times 3!}+\frac{2}{2!\times 2!\times 2!\times 2!}+\frac{1}{2!\times 4!\times 2!}+\frac{2}{2!\times 3!\times 2!}+\right.\\
&\qquad\left.\frac{2}{2!\times 2!\times 3!\times 2!}+\frac{1}{2!\times 2!\times 2!\times 2!}\right)+\\
&\quad (d-3)\,(d-4)\,(d-5)\,12!\left(\frac{1}{2!\times 2!\times 2!\times 2!}+\frac{1}{3!\times 3!}+\frac{2}{2!\times 2!\times 3!}\right)+\\
&\quad \frac{(d-3)\,(d-4)\,(d-5)\,(d-6)}{4!}\,\frac{12!}{2!};
\end{aligned}
$$

(* p_n($e_1+e_2+e_3+e_4$) *)

$$\text{NumberOfSRWTox}[\{4\},4] = 4!;$$

$$\text{NumberOfSRWTox}[\{4\},6] = 6!\,\frac{4}{2!}+(d-4)\,6!;$$

$$
\begin{aligned}
&\text{NumberOfSRWTox}[\{4\},8] = \\
&\quad 8!\left(\frac{4}{3!\times 2!}+\frac{4!}{2!\times 2!}\,\frac{1}{2!\times 2!}\right)+(d-4)\,8!\left(\frac{4}{2!}+\frac{1}{2!\times 2!}\right)+(d-4)\,\frac{(d-5)}{2}\,8!;
\end{aligned}
$$

$$
\begin{aligned}
&\text{NumberOfSRWTox}[\{4\},10] = 10!\left(\frac{4}{4!\times 3!}+4*3*\frac{1}{3!\times 2!\times 2!}+4\,\frac{1}{(2!)^3}\right)+\\
&\quad (d-4)\,10!\left(\frac{4}{3!\times 2!}+\frac{4!}{2!\times 2!}\,\frac{1}{2!\times 2!}+\frac{4}{2!\times 2!\times 2!}+\frac{1}{3!\times 3!}\right)+\\
&\quad (d-4)\,(d-5)\,10!\left(\frac{4}{2!\times 2!}+\frac{1}{2!\times 2!}\right)+\frac{(d-4)\,(d-5)\,(d-6)}{3!}\,10!;
\end{aligned}
$$

$$
\begin{aligned}
&\text{NumberOfSRWTox}[\{4\},12] = \\
&\quad 12!\left(\frac{4}{5!\times 4!}+4*3\,\frac{1}{4!\times 3!\times 2!}+\frac{4*3}{2}\,\frac{1}{(3!\times 2!)^2}+4*\frac{3*2}{2}\,\frac{1}{3!\times 2!\times 2!\times 2!}+\frac{1}{(2!)^4}\right)+\\
&\quad (d-4)\,12!\left(\frac{1}{4!\times 4!}+\frac{4}{3!\times 3!\times 2!}+\frac{4}{2!\times 2!\times 3!\times 2!}+\frac{4*3}{2}\,\frac{1}{(2!)^4}+4\,\frac{1}{4!\times 3!}+\right.\\
&\qquad\left.4*3\,\frac{1}{3!\times 2!\times 2!}+\frac{4}{(2!)^3}\right)+\\
&\quad (d-4)\,(d-5)\,12!\left(\frac{1}{3!\times 3!}+\frac{1}{(2!)^5}+\frac{4}{2!\times 2!\times 2!}+\frac{4}{3!\times 2!\times 2!}+\frac{4*3}{2}\,\frac{1}{2!\times 2!\times 2!}\right)+\\
&\quad (d-4)\,(d-5)\,(d-6)\,12!\left(\frac{1}{(2!)^3}+\frac{4}{3!\times 2!}\right)+\frac{(d-4)\,(d-5)\,(d-6)\,(d-7)}{4!}\,12!;
\end{aligned}
$$

1.3 Computation of $I_{n,0}(x)$

$$I_{n,0}(x) = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} \prod_{i=1}^d \int_{-\pi}^{\pi} e^{-t/d(1-\cos(k_i))} e^{\mathrm{t} k_i x_i} \frac{\mathrm{dk}}{2\pi} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} \prod_{i=1}^d F(t, d, |x_i|)$$

where F(t,d,n) is the modified Bessel function.

```
F[t_, d_, N_] := e^(-t/d) BesselI[N, t/d];
NIntproto[n_, d_, T_] :=
  1 / ((n - 1) !) * NIntegrate[t ^ (n - 1) * (F[t, d, 0]) ^ (d), {t, 0, T},
    WorkingPrecision → 20];
NIntproto2[n_, d_, T_, vecn_] :=
  1 / ((n - 1) !) *
   NIntegrate[t ^ (n - 1) * (F[t, d, 0]) ^ (d - Total[vecn]) *
     Product[F[t, d, i]^vecn[[i]], {i, 1, Length[vecn]}], {t, 0, T},
    WorkingPrecision → 20];
SRWInt[n_, dim_, vecn_] := If[vecn == {0}, NIntproto[n, dim, ∞],
   NIntproto2[n, dim, ∞, vecn]];
Clear[i];
```

Remark: We have chosen the representation of the point v(x), such that the above computation becomes as simple as it is now.

1.4 Computation of $I_{n,l}(x)$

Next, we compute the values of $I_{n,l}(x)$ for the different x using $I_{n,l}(x) = I_{n,(l-1)}(x) - I_{(n-1),(l-1)}(x)$

```
Do[
 tabletmp = Table[0, {s, 1, MaxNumberOfSteps + 2}, {t, 1, 2 + param}];
 (*creating an empty table*)
 tabletmp[[1, 1]] = Text["l \\ n"]; (*formatting top/left entry*)
 Do[tabletmp[[1, n + 2]] = n, {n, 0, param}]; (*formatting first row of table*)
 Do[tabletmp[[l + 2, 1]] = l, {l, 0, MaxNumberOfSteps}];
 (*formatting first coloumn of table*)
 Do[tabletmp[[2, n + 2]] = SRWInt[n, d, v];, {n, 1, param}];
 (*compute second row*)
 Do[tabletmp[[l + 2, 2]] = NumberOfSRWTox[v, l] / (2 d)^l;, {l, 0, MaxNumberOfSteps}];
 (*fill second coloumn*)
 Do[
  Do[tabletmp[[l + 3, n + 2]] = tabletmp[[l + 2, n + 2]] - tabletmp[[l + 2, n + 1]], {n, 1, param}];
  (*fill the rest of the table using I_n,l (x)=I_n,(l-1) (x)-I_(n-1),(l-1) (x)*)
  , {l, 0, MaxNumberOfSteps - 1}];
 SRWTwoPointFunctionTable[v] = tabletmp;
 Clear[n, l, s, t, tabletmp];
 , {v, NVecAll}]
```

1.5 Print-Out of $I_{n,l}(x)$

We print out the computed value of the SRW-Integrals. At the moment this part is deactivated to save the paper when printed.

```
(*Do[
 NForm[a_]:=NumberForm[N[a] ,5];
 OutputTable=Map[NForm,SRWTwoPointFunctionTable[v],{2}];
 Print[
   Labeled[Grid[OutputTable,
      Alignment → {{Left, Center}, Baseline, {{2, 14}, {2, param+2}} → {"."}},
      Frame → True, Dividers → {{2 → True, -1 → True}, {2 → True}},
      Spacings → {1.5, {1.5, 1, {0.5}}}, ItemStyle → {1 → Bold, 1 → Bold},
      Background → {Automatic, Automatic,
        {{2, 14}, {2, param+2}} → GrayLevel[0.9]}],
     Style["Value of the SRW two-point function in dimension "Text[d]Text[v],
      Bold], Top] // Text]
  Clear[OutputTable];
 ,{v,NVecAll}]*)
```

For better readability of the code we define the following function to access the values of $I_{n,l}(x)$:

```
Ivalue[n_, l_, v_] :=
  Module[ {value},
   value = SRWTwoPointFunctionTable[v][[l + 2, n + 2]];
   value
  ];
```

2. Other SRW-Integrals

Now we use the computed values of $I_{n,l}(x)$ to bound the other SRW integrals, see Section 5.2. We begin with $L_n(x)$ which we can compute explicitly, see (5.7) and (5.17).

$L_n(x)$

```
Do[
 L[n, {1}] = 1/(2 d) Ivalue[n, 0, {0}] + 1/(2 d) Ivalue[n, 0, {0, 1}] + (d - 1)/d Ivalue[n, 0, {2}];
 L[n, {0, 1}] = 1/(2 d) Ivalue[n, 0, {0}] + 1/(2 d) Ivalue[n, 0, {0, 0, 0, 1}] +
   (d - 1)/d Ivalue[n, 0, {0, 2}];
 L[n, {2}] = ((d - 2) (d - 3))/(d (d - 1)) Ivalue[n, 0, {4}] +
   (d - 2)/(2 d (d - 1)) (Ivalue[n, 0, {2}] + Ivalue[n, 0, {2, 1}]) +
   1/(4 d (d - 1)) (Ivalue[n, 0, {0}] + Ivalue[n, 0, {0, 2}] + 2 Ivalue[n, 0, {0, 1}]);
 , {n, 0, param}]
```

$K_{n,l}(x)$

Then we bound $K_{n,l}(x)$ as described in (3.36), and then improve the bound for $x = e_1$, $e_1 + e_2$, $2\,e_1$ using (5.18) and (5.19).

```
Do[Do[
    K[n, l, {1}] = Min[Ivalue[n, 2 l, {0}]^(1/2) L[n, {1}]^(1/2), Ivalue[n, l + 1, {0}]];
    K[n, l, {0, 1}] = Ivalue[n, 2 l, {0}]^(1/2) L[n, {0, 1}]^(1/2);
    K[n, l, {2}] = Ivalue[n, 2 l, {0}]^(1/2) L[n, {2}]^(1/2);
    , {l, 0, 6}] ×
   Do[
    K[n, l, {0, 1}] = Ivalue[n, 12, {0}]^(1/2) L[n, {0, 1}]^(1/2);
    K[n, l, {2}] = Ivalue[n, 12, {0}]^(1/2) L[n, {2}]^(1/2);
    , {l, 7, 12}] ×
   Do[
    K[n, l, {1}] = Min[Ivalue[n, 12, {0}]^(1/2) L[n, {1}]^(1/2), Ivalue[n, l + 1, {0}]];
    , {l, 7, 11}];
  K[n, 12, {1}] = Min[Ivalue[n, 12, {0}]^(1/2) L[n, {1}]^(1/2), Ivalue[n, 12, {0}]];
  , {n, 0, param}];

Do[Do[
   K[n, 0, v] = K[n - 1, 0, v] + Ivalue[n - 1, 4, {0}]^(1/2) L[n - 1, v]^(1/2) +
      Ivalue[n, 4, {0}]^(1/2) L[n, v]^(1/2);
   , {n, 1, param}], {v, {{1}, {0, 1}, {2}}}];
```

For $x = 0$ we bound use the bound given in (5.14):

```
Do[
  Do[
   K[n, 2 l, {0}] = Ivalue[n, 2 l, {0}];
   , {l, 0, 6}];
  Do[
   K[n, 2 l + 1, {0}] = Sqrt[Ivalue[n, 2 l, {0}] × Ivalue[n, 2 l + 2, {0}]];
   , {l, 0, 5}];
  , {n, 0, param}];
```

$T_{n,l}(x)$

We bound $T_{n,l}(x)$ as defined in (3.37) and bounded in (5.11):

```
Do[Do[Do[
    T[n, l, v] = K[n, l + 1, v] + Min[4/d K[n, l, v], 2/d K[n + 1, l, v]];
    , {n, 1, param - 1}];
   T[param, l, v] = K[param, l + 1, v] + 4/d K[param, l, v];
   , {l, 0, 12}], {v, {{0}, {1}, {0, 1}, {2}}}];
```

$V_{n,l}$

We compute the bounds on $V_{n,l}$ defined in (3.37) and (5.8), and computed in (5.25):

```
Do[Do[
   V[n, l] =
     1/(2 d)^2 (Ivalue[n, l, {0}] - 2 Ivalue[n, l, {0, 1}] + (d - 1)/d Ivalue[n, l, {0, 2}] +
        1/(2 d) Ivalue[n, l, {0}] + 1/(2 d) Ivalue[n, l, {0, 0, 0, 1}]);
   , {n, 0, param}]
  , {l, 0, 12}];
```

$U_{n,l}(x)$

Then we compute $U_{n,l}(x)$ as given in (3.38) and bounded in (5.9) and (5.12)-(5.13):

```
Do[Do[Do[
     U[n, l, v] = V[n, 2 l]^(1/2) L[n, v]^(1/2);
     , {v, {{0, 1}, {2}}}];
   If[Mod[l, 2] == 0, U[n, l, {0}] = 1/(2 d) (Ivalue[n, l, {0}] - Ivalue[n, l, {0, 1}]),
     U[n, l, {0}] = 1/d K[n, l, {0}]];
   , {n, 0, param}], {l, 0, 6}];

Do[Do[Do[
     U[n, l, v] = V[n, 12]^(1/2) L[n, v]^(1/2);
     , {v, {{0, 1}, {2}}}];
   U[n, l, {0}] = U[n, 6, {0}];
   , {n, 0, param}], {l, 7, 12}];

Do[Do[
   U[n, l, {1}] = Min[V[n, 2 l]^(1/2) L[n, {1}]^(1/2), U[n, l + 1, {0}]];
   , {l, 0, 5}];
  U[n, 6, {1}] = V[n, 12]^(1/2) L[n, {1}]^(1/2);
  , {n, 0, param}];
```

Weighted SRW-diagram

We begin with the bound for the initial point, derived in Section 2.3.3, see (2.30).

```
IM[n_, l_, v_] := Ivalue[n + 2, l + 1, v] - 1/d Ivalue[n + 3, l, v] +
   1/(2 d^2) Switch[v, {0}, 2 d Ivalue[n + 3, l, {0, 1}], {1},
     Ivalue[n + 3, l, {1}] + Ivalue[n + 3, l, {0, 0, 1}] + (2 d - 2) Ivalue[n + 3, l, {1, 1}],
     {0, 1}, Ivalue[n + 3, l, {0}] + Ivalue[n + 3, l, {0, 0, 0, 1}] +
      (2 d - 2) Ivalue[n + 3, l, {0, 2}], {2},
     2 Ivalue[n + 3, l, {2}] + 2 Ivalue[n + 3, l, {1, 0, 1}] +
      (2 d - 4) Ivalue[n + 3, l, {2, 1}]];
```

We remark at this point again that in this basic case only the case S={0} and S= $Z^d/\{0\}$ can be considered.

Print-out of the used values

We print out the value that we use for our computations:

```
(*TableSup=Table[0,{s,1,5},{t,1,4}];
n=2;l=4;
TableSup[[1,2]]={1};TableSup[[1,3]]={0,1};TableSup[[1,4]]={2};
TableSup[[2,1]]=Text[Lnl];
TableSup[[2,2]]=L[n,{1}];
TableSup[[2,3]]=L[n,{0,1}];
TableSup[[2,4]]=L[n,{2}];
TableSup[[3,1]]=Text[Knl];
TableSup[[3,2]]=K[n,l,{1}];
TableSup[[3,3]]=K[n,l,{0,1}];
TableSup[[3,4]]=K[n,l,{2}];
TableSup[[4,1]]=Text[Unl];
TableSup[[4,2]]=U[n,l,{1}];
TableSup[[4,3]]=U[n,l,{0,1}];
TableSup[[4,4]]=U[n,l,{2}];
TableSup[[5,1]]=Text[Tnl];
TableSup[[5,2]]=T[n,l,{1}];
TableSup[[5,3]]=T[n,l,{0,1}];
TableSup[[5,4]]=T[n,l,{2}];
NForm[a_]:=NumberForm[N[a] ,6];
Print[
  Labeled[Grid[Map[NForm,TableSup,{2}],
     Alignment → {{Left, Center}, Baseline, {{2, 5}, {2, 4}} → {"."}},
     Frame → True, Dividers → {{2 → True, -1 → True}, {2 → True}},
     Spacings → {1.5, {1.5, 1, {0.5}}}, ItemStyle → {1 → Bold, 1 → Bold},
     Background → {Automatic, Automatic, {{2, 5}, {2, 4}} → GrayLevel[0.9]}],
    Style["Bounds of the SRW two-point function "Text[d], Bold], Top] // Text]
 Clear[n,l]*)
```

We define the following functions to access the value of the supremum of the functions for $x \neq 0$:

3. Number of Self-avoiding and Bond-avoiding walks

We improve our bounds on simple diagrams by explicitly extracting short contributions. For this we have computed the number of self-avoiding walks (SAW) and bond-avoiding walks (BAW) with a specific number of steps. As we use this for all model we save/store these numbers here in the model-independent part of the implementation. These value are computed using a simple JAVA program that is available on the website of the author.

```
ComputedPoints = Join[Table[{i}, {i, 0, 10}], Table[{i, 1}, {i, 0, 8}],
   Table[{i, 0, 1}, {i, 0, 7}], Table[{i, 0, 0, 1}, {i, 0, 6}],
   Table[{i, 0, 0, 0, 1}, {i, 0, 5}], Table[{i, 0, 0, 0, 0, 1}, {i, 0, 4}],
   Table[{i, 0, 0, 0, 0, 0, 1}, {i, 0, 3}], Table[{i, 0, 0, 0, 0, 0, 0, 1}, {i, 0, 2}],
   Table[{i, 0, 0, 0, 0, 0, 0, 0, 1}, {i, 0, 1}], Table[{i, 2}, {i, 0, 6}],
   Table[{i, 0, 2}, {i, 0, 4}], Table[{i, 0, 0, 2}, {i, 0, 2}], Table[{i, 3}, {i, 0, 4}],
   Table[{i, 0, 3}, {i, 0, 1}], Table[{i, 4}, {i, 0, 2}], Table[{i, 5}, {i, 0, 0}],
   Table[{i, 1, 1}, {i, 0, 5}]];
```

```
ComputedSteps = 10;
Do[Do[
   nrSAW[n, d, v] = 0;
```

```
    nrBAW[n, d, v] = 0;, {n, 0, ComputedSteps}], {v, ComputedPoints}];

(*n=1,SAW*)
nrSAW[1, d, {1}] = If[(d ≥ 1), 1, 0];
(*n=1,BAW*)
nrBAW[1, d, {1}] = If[(d ≥ 1), 1, 0];
(*n=2,SAW*)
nrSAW[2, d, {0}] = If[(d ≥ 1), 1, 0];
nrSAW[2, d, {2}] = If[(d ≥ 2), 2, 0];
(*n=2,BAW*)
nrBAW[2, d, {0}] = If[(d ≥ 1), 1, 0];
nrBAW[2, d, {2}] = If[(d ≥ 2), 2, 0];

(*n=3,SAW*)
nrSAW[3, d, {1}] = If[(d ≥ 1), 2 (d - 1), 0];
nrSAW[3, d, {0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrSAW[3, d, {1, 1}] = If[(d ≥ 2), 3, 0];
nrSAW[3, d, {3}] = If[(d ≥ 3), 6, 0];
(*n=3,BAW*)
nrBAW[3, d, {1}] = If[(d ≥ 1), 2 (d - 1), 0];
nrBAW[3, d, {0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrBAW[3, d, {1, 1}] = If[(d ≥ 2), 3, 0];
nrBAW[3, d, {3}] = If[(d ≥ 3), 6, 0];

(*n=4,SAW*)
nrSAW[4, d, {2}] = If[(d ≥ 2), 4 + 12 (d - 2), 0];
nrSAW[4, d, {0, 1}] = If[(d ≥ 1), 6 (d - 1), 0];
nrSAW[4, d, {0, 0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrSAW[4, d, {1, 0, 1}] = If[(d ≥ 2), 4, 0];
nrSAW[4, d, {0, 2}] = If[(d ≥ 2), 6, 0];
nrSAW[4, d, {2, 1}] = If[(d ≥ 3), 12, 0];
nrSAW[4, d, {4}] = If[(d ≥ 4), 24, 0];
(*n=4,BAW*)
nrBAW[4, d, {2}] = If[(d ≥ 2), 4 + 12 (d - 2), 0];
nrBAW[4, d, {0, 1}] = If[(d ≥ 1), 6 (d - 1), 0];
nrBAW[4, d, {0}] = If[(d ≥ 0), 4 d (d - 1), 0];
nrBAW[4, d, {0, 0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrBAW[4, d, {1, 0, 1}] = If[(d ≥ 2), 4, 0];
nrBAW[4, d, {0, 2}] = If[(d ≥ 2), 6, 0];
nrBAW[4, d, {2, 1}] = If[(d ≥ 3), 12, 0];
nrBAW[4, d, {4}] = If[(d ≥ 4), 24, 0];

(*n=5,SAW*)
nrSAW[5, d, {3}] = If[(d ≥ 3), 54 + 72 (d - 3), 0];
nrSAW[5, d, {1, 1}] = If[(d ≥ 2), 11 + 36 (d - 2), 0];
nrSAW[5, d, {1}] = If[(d ≥ 1), 6 (d - 1) + 16 (d - 1) (d - 2), 0];
nrSAW[5, d, {0, 0, 1}] = If[(d ≥ 1), 12 (d - 1), 0];
nrSAW[5, d, {0, 0, 0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrSAW[5, d, {1, 0, 0, 1}] = If[(d ≥ 2), 5, 0];
nrSAW[5, d, {0, 1, 1}] = If[(d ≥ 2), 10, 0];
nrSAW[5, d, {2, 0, 1}] = If[(d ≥ 3), 20, 0];
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nrSAW[5, d, {1, 2}] = If[(d ≥ 3), 30, 0];
nrSAW[5, d, {3, 1}] = If[(d ≥ 4), 60, 0];
nrSAW[5, d, {5}] = If[(d ≥ 5), 120, 0];
(*n=5,BAW*)
nrBAW[5, d, {3}] = If[(d ≥ 3), 54 + 72 (d - 3), 0];
nrBAW[5, d, {1, 1}] = If[(d ≥ 2), 11 + 36 (d - 2), 0];
nrBAW[5, d, {1}] = If[(d ≥ 1), 14 (d - 1) + 24 (d - 1) (d - 2), 0];
nrBAW[5, d, {0, 0, 1}] = If[(d ≥ 1), 12 (d - 1), 0];
nrBAW[5, d, {0, 0, 0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrBAW[5, d, {1, 0, 0, 1}] = If[(d ≥ 2), 5, 0];
nrBAW[5, d, {0, 1, 1}] = If[(d ≥ 2), 10, 0];
nrBAW[5, d, {2, 0, 1}] = If[(d ≥ 3), 20, 0];
nrBAW[5, d, {1, 2}] = If[(d ≥ 3), 30, 0];
nrBAW[5, d, {3, 1}] = If[(d ≥ 4), 60, 0];
nrBAW[5, d, {5}] = If[(d ≥ 5), 120, 0];
(*n=6,SAW*)
nrSAW[6, d, {4}] = If[(d ≥ 4), 576 + 480 (d - 4), 0];
nrSAW[6, d, {2, 1}] = If[(d ≥ 3), 168 + 240 (d - 3), 0];
nrSAW[6, d, {2}] = If[(d ≥ 2), 16 + 168 (d - 2) + 144 (d - 2) (d - 3), 0];
nrSAW[6, d, {1, 0, 1}] = If[(d ≥ 2), 26 + 80 (d - 2), 0];
nrSAW[6, d, {0, 2}] = If[(d ≥ 2), 24 + 120 (d - 2), 0];
nrSAW[6, d, {0, 1}] = If[(d ≥ 1), 20 (d - 1) + 72 (d - 1) (d - 2), 0];
nrSAW[6, d, {0, 0, 0, 1}] = If[(d ≥ 1), 20 (d - 1), 0];
nrSAW[6, d, {0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrSAW[6, d, {1, 0, 0, 0, 1}] = If[(d ≥ 2), 6, 0];
nrSAW[6, d, {0, 1, 0, 1}] = If[(d ≥ 2), 15, 0];
nrSAW[6, d, {2, 0, 0, 1}] = If[(d ≥ 3), 30, 0];
nrSAW[6, d, {0, 0, 2}] = If[(d ≥ 2), 20, 0];
nrSAW[6, d, {1, 1, 1}] = If[(d ≥ 3), 60, 0];
nrSAW[6, d, {3, 0, 1}] = If[(d ≥ 4), 120, 0];
nrSAW[6, d, {0, 3}] = If[(d ≥ 3), 90, 0];
nrSAW[6, d, {2, 2}] = If[(d ≥ 4), 180, 0];
nrSAW[6, d, {4, 1}] = If[(d ≥ 5), 360, 0];
nrSAW[6, d, {6}] = If[(d ≥ 6), 720, 0];
(*n=6,BAW*)
nrBAW[6, d, {4}] = If[(d ≥ 4), 576 + 480 (d - 4), 0];
nrBAW[6, d, {2, 1}] = If[(d ≥ 3), 168 + 240 (d - 3), 0];
nrBAW[6, d, {2}] = If[(d ≥ 2), 36 + 232 (d - 2) + 168 (d - 2) (d - 3), 0];
nrBAW[6, d, {1, 0, 1}] = If[(d ≥ 2), 26 + 80 (d - 2), 0];
nrBAW[6, d, {0, 2}] = If[(d ≥ 2), 24 + 120 (d - 2), 0];
nrBAW[6, d, {0, 1}] = If[(d ≥ 1), 28 (d - 1) + 84 (d - 1) (d - 2), 0];
nrBAW[6, d, {0, 0, 0, 1}] = If[(d ≥ 1), 20 (d - 1), 0];
nrBAW[6, d, {0}] = If[(d ≥ 0), 12 d (d - 1) + 32 d (d - 1) (d - 2), 0];
nrBAW[6, d, {0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrBAW[6, d, {1, 0, 0, 0, 1}] = If[(d ≥ 2), 6, 0];
nrBAW[6, d, {0, 1, 0, 1}] = If[(d ≥ 2), 15, 0];
nrBAW[6, d, {2, 0, 0, 1}] = If[(d ≥ 3), 30, 0];
nrBAW[6, d, {0, 0, 2}] = If[(d ≥ 2), 20, 0];
nrBAW[6, d, {1, 1, 1}] = If[(d ≥ 3), 60, 0];
nrBAW[6, d, {3, 0, 1}] = If[(d ≥ 4), 120, 0];
nrBAW[6, d, {0, 3}] = If[(d ≥ 3), 90, 0];
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nrBAW[6, d, {2, 2}] = If[(d ≥ 4), 180, 0];
nrBAW[6, d, {4, 1}] = If[(d ≥ 5), 360, 0];
nrBAW[6, d, {6}] = If[(d ≥ 6), 720, 0];

(*n=7,SAW*)
nrSAW[7, d, {5}] = If[(d ≥ 5), 6000 + 3600 (d - 5), 0];
nrSAW[7, d, {3, 1}] = If[(d ≥ 4), 2040 + 1800 (d - 4), 0];
nrSAW[7, d, {3}] = If[(d ≥ 3), 708 + 2520 (d - 3) + 1224 (d - 3) (d - 4), 0];
nrSAW[7, d, {2, 0, 1}] = If[(d ≥ 3), 430 + 600 (d - 3), 0];
nrSAW[7, d, {1, 2}] = If[(d ≥ 3), 540 + 900 (d - 3), 0];
nrSAW[7, d, {1, 1}] = If[(d ≥ 2), 45 + 636 (d - 2) + 612 (d - 2) (d - 3), 0];
nrSAW[7, d, {1, 0, 0, 1}] = If[(d ≥ 2), 52 + 150 (d - 2), 0];
nrSAW[7, d, {0, 1, 1}] = If[(d ≥ 2), 55 + 300 (d - 2), 0];
nrSAW[7, d, {1}] = If[(d ≥ 1), 28 (d - 1) + 248 (d - 1) (d - 2) + 216 (d - 1) (d - 2) (d - 3),
   0];
nrSAW[7, d, {0, 0, 1}] = If[(d ≥ 1), 54 (d - 1) + 204 (d - 1) (d - 2), 0];
nrSAW[7, d, {0, 0, 0, 0, 1}] = If[(d ≥ 1), 30 (d - 1), 0];
nrSAW[7, d, {0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrSAW[7, d, {1, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 7, 0];
nrSAW[7, d, {0, 1, 0, 0, 1}] = If[(d ≥ 2), 21, 0];
nrSAW[7, d, {2, 0, 0, 0, 1}] = If[(d ≥ 3), 42, 0];
nrSAW[7, d, {0, 0, 1, 1}] = If[(d ≥ 2), 35, 0];
nrSAW[7, d, {1, 1, 0, 1}] = If[(d ≥ 3), 105, 0];
nrSAW[7, d, {3, 0, 0, 1}] = If[(d ≥ 4), 210, 0];
nrSAW[7, d, {1, 0, 2}] = If[(d ≥ 3), 140, 0];
nrSAW[7, d, {0, 2, 1}] = If[(d ≥ 3), 210, 0];
nrSAW[7, d, {2, 1, 1}] = If[(d ≥ 4), 420, 0];
nrSAW[7, d, {4, 0, 1}] = If[(d ≥ 5), 840, 0];
nrSAW[7, d, {1, 3}] = If[(d ≥ 4), 630, 0];
nrSAW[7, d, {3, 2}] = If[(d ≥ 5), 1260, 0];
nrSAW[7, d, {5, 1}] = If[(d ≥ 6), 2520, 0];
nrSAW[7, d, {7}] = If[(d ≥ 7), 5040, 0];
(*n=7,BAW*)
nrBAW[7, d, {5}] = If[(d ≥ 5), 6000 + 3600 (d - 5), 0];
nrBAW[7, d, {3, 1}] = If[(d ≥ 4), 2040 + 1800 (d - 4), 0];
nrBAW[7, d, {3}] = If[(d ≥ 3), 1020 + 2952 (d - 3) + 1320 (d - 3) (d - 4), 0];
nrBAW[7, d, {2, 0, 1}] = If[(d ≥ 3), 430 + 600 (d - 3), 0];
nrBAW[7, d, {1, 2}] = If[(d ≥ 3), 540 + 900 (d - 3), 0];
nrBAW[7, d, {1, 1}] = If[(d ≥ 2), 77 + 756 (d - 2) + 660 (d - 2) (d - 3), 0];
nrBAW[7, d, {1, 0, 0, 1}] = If[(d ≥ 2), 52 + 150 (d - 2), 0];
nrBAW[7, d, {0, 1, 1}] = If[(d ≥ 2), 55 + 300 (d - 2), 0];
nrBAW[7, d, {1}] = If[(d ≥ 1), 70 (d - 1) + 468 (d - 1) (d - 2) + 312 (d - 1) (d - 2) (d - 3),
   0];
nrBAW[7, d, {0, 0, 1}] = If[(d ≥ 1), 62 (d - 1) + 220 (d - 1) (d - 2), 0];
nrBAW[7, d, {0, 0, 0, 0, 1}] = If[(d ≥ 1), 30 (d - 1), 0];
nrBAW[7, d, {0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrBAW[7, d, {1, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 7, 0];
nrBAW[7, d, {0, 1, 0, 0, 1}] = If[(d ≥ 2), 21, 0];
nrBAW[7, d, {2, 0, 0, 0, 1}] = If[(d ≥ 3), 42, 0];
nrBAW[7, d, {0, 0, 1, 1}] = If[(d ≥ 2), 35, 0];
nrBAW[7, d, {1, 1, 0, 1}] = If[(d ≥ 3), 105, 0];
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nrBAW[7, d, {3, 0, 0, 1}] = If[(d ≥ 4), 210, 0];
nrBAW[7, d, {1, 0, 2}] = If[(d ≥ 3), 140, 0];
nrBAW[7, d, {0, 2, 1}] = If[(d ≥ 3), 210, 0];
nrBAW[7, d, {2, 1, 1}] = If[(d ≥ 4), 420, 0];
nrBAW[7, d, {4, 0, 1}] = If[(d ≥ 5), 840, 0];
nrBAW[7, d, {1, 3}] = If[(d ≥ 4), 630, 0];
nrBAW[7, d, {3, 2}] = If[(d ≥ 5), 1260, 0];
nrBAW[7, d, {5, 1}] = If[(d ≥ 6), 2520, 0];
nrBAW[7, d, {7}] = If[(d ≥ 7), 5040, 0];


(*n=8,SAW*)
nrSAW[8, d, {6}] = If[(d ≥ 6), 64800 + 30240 (d - 6), 0];
nrSAW[8, d, {4, 1}] = If[(d ≥ 5), 24000 + 15120 (d - 5), 0];
nrSAW[8, d, {4}] = If[(d ≥ 4), 16296 + 32976 (d - 4) + 11040 (d - 4) (d - 5), 0];
nrSAW[8, d, {3, 0, 1}] = If[(d ≥ 4), 5760 + 5040 (d - 4), 0];
nrSAW[8, d, {2, 2}] = If[(d ≥ 4), 7800 + 7560 (d - 4), 0];
nrSAW[8, d, {2, 1}] = If[(d ≥ 3), 2586 + 10488 (d - 3) + 5520 (d - 3) (d - 4), 0];
nrSAW[8, d, {2, 0, 0, 1}] = If[(d ≥ 3), 936 + 1260 (d - 3), 0];
nrSAW[8, d, {1, 1, 1}] = If[(d ≥ 3), 1480 + 2520 (d - 3), 0];
nrSAW[8, d, {0, 3}] = If[(d ≥ 3), 1800 + 3780 (d - 3), 0];
nrSAW[8, d, {2}] =
  If[(d ≥ 2), 76 + 2468 (d - 2) + 6192 (d - 2) (d - 3) + 2480 (d - 2) (d - 3) (d - 4), 0];
nrSAW[8, d, {1, 0, 1}] = If[(d ≥ 2), 118 + 1916 (d - 2) + 1840 (d - 2) (d - 3), 0];
nrSAW[8, d, {0, 2}] = If[(d ≥ 2), 112 + 2244 (d - 2) + 2760 (d - 2) (d - 3), 0];
nrSAW[8, d, {1, 0, 0, 0, 1}] = If[(d ≥ 2), 92 + 252 (d - 2), 0];
nrSAW[8, d, {0, 1, 0, 1}] = If[(d ≥ 2), 118 + 630 (d - 2), 0];
nrSAW[8, d, {0, 0, 2}] = If[(d ≥ 2), 120 + 840 (d - 2), 0];
nrSAW[8, d, {0, 1}] =
  If[(d ≥ 1), 92 (d - 1) + 1216 (d - 1) (d - 2) + 1240 (d - 1) (d - 2) (d - 3), 0];
nrSAW[8, d, {0, 0, 0, 1}] = If[(d ≥ 1), 126 (d - 1) + 460 (d - 1) (d - 2), 0];
nrSAW[8, d, {0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 42 (d - 1), 0];
nrSAW[8, d, {0, 0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrSAW[8, d, {1, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 8, 0];
nrSAW[8, d, {0, 1, 0, 0, 0, 1}] = If[(d ≥ 2), 28, 0];
nrSAW[8, d, {2, 0, 0, 0, 0, 1}] = If[(d ≥ 3), 56, 0];
nrSAW[8, d, {0, 0, 1, 0, 1}] = If[(d ≥ 2), 56, 0];
nrSAW[8, d, {1, 1, 0, 0, 1}] = If[(d ≥ 3), 168, 0];
nrSAW[8, d, {3, 0, 0, 0, 1}] = If[(d ≥ 4), 336, 0];
nrSAW[8, d, {0, 0, 0, 2}] = If[(d ≥ 2), 70, 0];
nrSAW[8, d, {1, 0, 1, 1}] = If[(d ≥ 3), 280, 0];
nrSAW[8, d, {0, 2, 0, 1}] = If[(d ≥ 3), 420, 0];
nrSAW[8, d, {2, 1, 0, 1}] = If[(d ≥ 4), 840, 0];
nrSAW[8, d, {4, 0, 0, 1}] = If[(d ≥ 5), 1680, 0];
nrSAW[8, d, {0, 1, 2}] = If[(d ≥ 3), 560, 0];
nrSAW[8, d, {2, 0, 2}] = If[(d ≥ 4), 1120, 0];
nrSAW[8, d, {1, 2, 1}] = If[(d ≥ 4), 1680, 0];
nrSAW[8, d, {3, 1, 1}] = If[(d ≥ 5), 3360, 0];
nrSAW[8, d, {5, 0, 1}] = If[(d ≥ 6), 6720, 0];
nrSAW[8, d, {0, 4}] = If[(d ≥ 4), 2520, 0];
nrSAW[8, d, {2, 3}] = If[(d ≥ 5), 5040, 0];
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nrSAW[8, d, {4, 2}] = If[(d ≥ 6), 10 080, 0];
nrSAW[8, d, {6, 1}] = If[(d ≥ 7), 20 160, 0];
nrSAW[8, d, {8}] = If[(d ≥ 8), 40 320, 0];
(*n=8,BAW*)
nrBAW[8, d, {6}] = If[(d ≥ 6), 64 800 + 30 240 (d - 6), 0];
nrBAW[8, d, {4, 1}] = If[(d ≥ 5), 24 000 + 15 120 (d - 5), 0];
nrBAW[8, d, {4}] = If[(d ≥ 4), 19 896 + 36 048 (d - 4) + 11 520 (d - 4) (d - 5), 0];
nrBAW[8, d, {3, 0, 1}] = If[(d ≥ 4), 5760 + 5040 (d - 4), 0];
nrBAW[8, d, {2, 2}] = If[(d ≥ 4), 7800 + 7560 (d - 4), 0];
nrBAW[8, d, {2, 1}] = If[(d ≥ 3), 3318 + 11 544 (d - 3) + 5760 (d - 3) (d - 4), 0];
nrBAW[8, d, {2, 0, 0, 1}] = If[(d ≥ 3), 936 + 1260 (d - 3), 0];
nrBAW[8, d, {1, 1, 1}] = If[(d ≥ 3), 1480 + 2520 (d - 3), 0];
nrBAW[8, d, {0, 3}] = If[(d ≥ 3), 1800 + 3780 (d - 3), 0];
nrBAW[8, d, {2}] =
  If[(d ≥ 2), 164 + 4140 (d - 2) + 8248 (d - 2) (d - 3) + 2912 (d - 2) (d - 3) (d - 4), 0];
nrBAW[8, d, {1, 0, 1}] = If[(d ≥ 2), 162 + 2108 (d - 2) + 1920 (d - 2) (d - 3), 0];
nrBAW[8, d, {0, 2}] = If[(d ≥ 2), 184 + 2532 (d - 2) + 2880 (d - 2) (d - 3), 0];
nrBAW[8, d, {1, 0, 0, 0, 1}] = If[(d ≥ 2), 92 + 252 (d - 2), 0];
nrBAW[8, d, {0, 1, 0, 1}] = If[(d ≥ 2), 118 + 630 (d - 2), 0];
nrBAW[8, d, {0, 0, 2}] = If[(d ≥ 2), 120 + 840 (d - 2), 0];
nrBAW[8, d, {0, 1}] =
  If[(d ≥ 1), 188 (d - 1) + 1676 (d - 1) (d - 2) + 1456 (d - 1) (d - 2) (d - 3), 0];
nrBAW[8, d, {0, 0, 0, 1}] = If[(d ≥ 1), 134 (d - 1) + 480 (d - 1) (d - 2), 0];
nrBAW[8, d, {0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 42 (d - 1), 0];
nrBAW[8, d, {0}] =
  If[(d ≥ 0), 88 d (d - 1) + 624 d (d - 1) (d - 2) + 496 d (d - 1) (d - 2) (d - 3), 0];
nrBAW[8, d, {0, 0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrBAW[8, d, {1, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 8, 0];
nrBAW[8, d, {0, 1, 0, 0, 0, 1}] = If[(d ≥ 2), 28, 0];
nrBAW[8, d, {2, 0, 0, 0, 0, 1}] = If[(d ≥ 3), 56, 0];
nrBAW[8, d, {0, 0, 1, 0, 1}] = If[(d ≥ 2), 56, 0];
nrBAW[8, d, {1, 1, 0, 0, 1}] = If[(d ≥ 3), 168, 0];
nrBAW[8, d, {3, 0, 0, 0, 1}] = If[(d ≥ 4), 336, 0];
nrBAW[8, d, {0, 0, 0, 2}] = If[(d ≥ 2), 70, 0];
nrBAW[8, d, {1, 0, 1, 1}] = If[(d ≥ 3), 280, 0];
nrBAW[8, d, {0, 2, 0, 1}] = If[(d ≥ 3), 420, 0];
nrBAW[8, d, {2, 1, 0, 1}] = If[(d ≥ 4), 840, 0];
nrBAW[8, d, {4, 0, 0, 1}] = If[(d ≥ 5), 1680, 0];
nrBAW[8, d, {0, 1, 2}] = If[(d ≥ 3), 560, 0];
nrBAW[8, d, {2, 0, 2}] = If[(d ≥ 4), 1120, 0];
nrBAW[8, d, {1, 2, 1}] = If[(d ≥ 4), 1680, 0];
nrBAW[8, d, {3, 1, 1}] = If[(d ≥ 5), 3360, 0];
nrBAW[8, d, {5, 0, 1}] = If[(d ≥ 6), 6720, 0];
nrBAW[8, d, {0, 4}] = If[(d ≥ 4), 2520, 0];
nrBAW[8, d, {2, 3}] = If[(d ≥ 5), 5040, 0];
nrBAW[8, d, {4, 2}] = If[(d ≥ 6), 10 080, 0];
nrBAW[8, d, {6, 1}] = If[(d ≥ 7), 20 160, 0];
nrBAW[8, d, {8}] = If[(d ≥ 8), 40 320, 0];

(*n=9, SAW*)
nrSAW[9, d, {7}] = If[(d ≥ 7), 740 880 + 282 240 (d - 7), 0];
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nrSAW[9, d, {5, 1}] = If[(d ≥ 6), 289 800 + 141 120 (d - 6), 0];
nrSAW[9, d, {5}] = If[(d ≥ 5), 305 520 + 424 800 (d - 5) + 108 000 (d - 5) (d - 6), 0];
nrSAW[9, d, {4, 0, 1}] = If[(d ≥ 5), 74 760 + 47 040 (d - 5), 0];
nrSAW[9, d, {3, 2}] = If[(d ≥ 5), 104 580 + 70 560 (d - 5), 0];
nrSAW[9, d, {3, 1}] = If[(d ≥ 4), 68 544 + 151 440 (d - 4) + 54 000 (d - 4) (d - 5), 0];
nrSAW[9, d, {3, 0, 0, 1}] = If[(d ≥ 4), 13 734 + 11 760 (d - 4), 0];
nrSAW[9, d, {2, 1, 1}] = If[(d ≥ 4), 23 940 + 23 520 (d - 4), 0];
nrSAW[9, d, {1, 3}] = If[(d ≥ 4), 32 130 + 35 280 (d - 4), 0];
nrSAW[9, d, {3}] =
  If[(d ≥ 3), 10 518 + 84 024 (d - 3) + 103 824 (d - 3) (d - 4) + 26 592 (d - 3) (d - 4) (d - 5),
   0];
nrSAW[9, d, {2, 0, 1}] = If[(d ≥ 3), 8146 + 34 080 (d - 3) + 18 000 (d - 3) (d - 4), 0];
nrSAW[9, d, {1, 2}] = If[(d ≥ 3), 9204 + 45 240 (d - 3) + 27 000 (d - 3) (d - 4), 0];
nrSAW[9, d, {2, 0, 0, 0, 1}] = If[(d ≥ 3), 1806 + 2352 (d - 3), 0];
nrSAW[9, d, {1, 1, 0, 1}] = If[(d ≥ 3), 3507 + 5880 (d - 3), 0];
nrSAW[9, d, {1, 0, 2}] = If[(d ≥ 3), 4340 + 7840 (d - 3), 0];
nrSAW[9, d, {0, 2, 1}] = If[(d ≥ 3), 5250 + 11 760 (d - 3), 0];
nrSAW[9, d, {1, 1}] =
  If[(d ≥ 2), 220 + 10 180 (d - 2) + 30 024 (d - 2) (d - 3) + 13 296 (d - 2) (d - 3) (d - 4), 0];
nrSAW[9, d, {1, 0, 0, 1}] = If[(d ≥ 2), 286 + 4812 (d - 2) + 4500 (d - 2) (d - 3), 0];
nrSAW[9, d, {0, 1, 1}] = If[(d ≥ 2), 277 + 6880 (d - 2) + 9000 (d - 2) (d - 3), 0];
nrSAW[9, d, {1, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 149 + 392 (d - 2), 0];
nrSAW[9, d, {0, 1, 0, 0, 1}] = If[(d ≥ 2), 231 + 1176 (d - 2), 0];
nrSAW[9, d, {0, 0, 1, 1}] = If[(d ≥ 2), 259 + 1960 (d - 2), 0];
nrSAW[9, d, {1}] =
  If[(d ≥ 1), 140 (d - 1) + 3880 (d - 1) (d - 2) + 9640 (d - 1) (d - 2) (d - 3) +
    3968 (d - 1) (d - 2) (d - 3) (d - 4), 0];
nrSAW[9, d, {0, 0, 1}] =
  If[(d ≥ 3), 256 (d - 1) + 4212 (d - 1) (d - 2) + 4432 (d - 1) (d - 2) (d - 3), 0];
nrSAW[9, d, {0, 0, 0, 0, 1}] = If[(d ≥ 2), 260 (d - 1) + 900 (d - 1) (d - 2), 0];
nrSAW[9, d, {0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 56 (d - 1), 0];
nrSAW[9, d, {1, 0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 9, 0];
nrSAW[9, d, {0, 1, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 36, 0];
nrSAW[9, d, {2, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 3), 72, 0];
nrSAW[9, d, {0, 0, 1, 0, 0, 1}] = If[(d ≥ 2), 84, 0];
nrSAW[9, d, {1, 1, 0, 0, 0, 1}] = If[(d ≥ 3), 252, 0];
nrSAW[9, d, {3, 0, 0, 0, 0, 1}] = If[(d ≥ 4), 504, 0];
nrSAW[9, d, {0, 0, 0, 1, 1}] = If[(d ≥ 2), 126, 0];
nrSAW[9, d, {1, 0, 1, 0, 1}] = If[(d ≥ 3), 504, 0];
nrSAW[9, d, {0, 2, 0, 0, 1}] = If[(d ≥ 3), 756, 0];
nrSAW[9, d, {2, 1, 0, 0, 1}] = If[(d ≥ 4), 1512, 0];
nrSAW[9, d, {4, 0, 0, 0, 1}] = If[(d ≥ 5), 3024, 0];
nrSAW[9, d, {1, 0, 0, 2}] = If[(d ≥ 3), 630, 0];
nrSAW[9, d, {0, 1, 1, 1}] = If[(d ≥ 3), 1260, 0];
nrSAW[9, d, {2, 0, 1, 1}] = If[(d ≥ 4), 2520, 0];
nrSAW[9, d, {1, 2, 0, 1}] = If[(d ≥ 4), 3780, 0];
nrSAW[9, d, {3, 1, 0, 1}] = If[(d ≥ 5), 7560, 0];
nrSAW[9, d, {5, 0, 0, 1}] = If[(d ≥ 6), 15 120, 0];
nrSAW[9, d, {0, 0, 3}] = If[(d ≥ 3), 1680, 0];
nrSAW[9, d, {1, 1, 2}] = If[(d ≥ 4), 5040, 0];
nrSAW[9, d, {3, 0, 2}] = If[(d ≥ 5), 10 080, 0];
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nrSAW[9, d, {0, 3, 1}] = If[(d ≥ 4), 7560, 0];
nrSAW[9, d, {2, 2, 1}] = If[(d ≥ 5), 15120, 0];
nrSAW[9, d, {4, 1, 1}] = If[(d ≥ 6), 30240, 0];
nrSAW[9, d, {6, 0, 1}] = If[(d ≥ 7), 60480, 0];
nrSAW[9, d, {1, 4}] = If[(d ≥ 5), 22680, 0];
nrSAW[9, d, {3, 3}] = If[(d ≥ 6), 45360, 0];
nrSAW[9, d, {5, 2}] = If[(d ≥ 7), 90720, 0];
nrSAW[9, d, {7, 1}] = If[(d ≥ 8), 181440, 0];
nrSAW[9, d, {9}] = If[(d ≥ 9), 362880, 0];

(*n=9, BAW*)
nrBAW[9, d, {7}] = If[(d ≥ 7), 740880 + 282240 (d - 7), 0];
nrBAW[9, d, {5, 1}] = If[(d ≥ 6), 289800 + 141120 (d - 6), 0];
nrBAW[9, d, {5}] = If[(d ≥ 5), 344880 + 448800 (d - 5) + 110880 (d - 5) (d - 6), 0];
nrBAW[9, d, {4, 0, 1}] = If[(d ≥ 5), 74760 + 47040 (d - 5), 0];
nrBAW[9, d, {3, 2}] = If[(d ≥ 5), 104580 + 70560 (d - 5), 0];
nrBAW[9, d, {3, 1}] = If[(d ≥ 4), 79056 + 160560 (d - 4) + 55440 (d - 4) (d - 5), 0];
nrBAW[9, d, {3, 0, 0, 1}] = If[(d ≥ 4), 13734 + 11760 (d - 4), 0];
nrBAW[9, d, {2, 1, 1}] = If[(d ≥ 4), 23940 + 23520 (d - 4), 0];
nrBAW[9, d, {1, 3}] = If[(d ≥ 4), 32130 + 35280 (d - 4), 0];
nrBAW[9, d, {3}] =
  If[(d ≥ 3), 17118 + 111996 (d - 3) + 121896 (d - 3) (d - 4) + 9088 (d - 3) (d - 4) (d - 5),
   0];
nrBAW[9, d, {2, 0, 1}] = If[(d ≥ 3), 9538 + 36160 (d - 3) + 18480 (d - 3) (d - 4), 0];
nrBAW[9, d, {1, 2}] = If[(d ≥ 3), 11316 + 48360 (d - 3) + 27720 (d - 3) (d - 4), 0];
nrBAW[9, d, {2, 0, 0, 0, 1}] = If[(d ≥ 3), 1806 + 2352 (d - 3), 0];
nrBAW[9, d, {1, 1, 0, 1}] = If[(d ≥ 3), 3507 + 5880 (d - 3), 0];
nrBAW[9, d, {1, 0, 2}] = If[(d ≥ 3), 4340 + 7840 (d - 3), 0];
nrBAW[9, d, {0, 2, 1}] = If[(d ≥ 3), 5250 + 11760 (d - 3), 0];
nrBAW[9, d, {1, 1}] =
  If[(d ≥ 2), 432 + 14662 (d - 2) + 35796 (d - 2) (d - 3) + 14544 (d - 2) (d - 3) (d - 4), 0];
nrBAW[9, d, {1, 0, 0, 1}] = If[(d ≥ 2), 342 + 5092 (d - 2) + 4620 (d - 2) (d - 3), 0];
nrBAW[9, d, {0, 1, 1}] = If[(d ≥ 2), 405 + 7440 (d - 2) + 9240 (d - 2) (d - 3), 0];
nrBAW[9, d, {1, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 149 + 392 (d - 2), 0];
nrBAW[9, d, {0, 1, 0, 0, 1}] = If[(d ≥ 2), 231 + 1176 (d - 2), 0];
nrBAW[9, d, {0, 0, 1, 1}] = If[(d ≥ 2), 259 + 1960 (d - 2), 0];
nrBAW[9, d, {1}] =
  If[(d ≥ 1), 408 (d - 1) + 8656 (d - 1) (d - 2) + 16712 (d - 1) (d - 2) (d - 3) +
    5616 (d - 1) (d - 2) (d - 3) (d - 4), 0];
nrBAW[9, d, {0, 0, 1}] =
  If[(d ≥ 1), 438 (d - 1) + 5088 (d - 1) (d - 2) + 4848 (d - 1) (d - 2) (d - 3), 0];
nrBAW[9, d, {0, 0, 0, 0, 1}] = If[(d ≥ 1), 268 (d - 1) + 924 (d - 1) (d - 2), 0];
nrBAW[9, d, {0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 56 (d - 1), 0];
nrBAW[9, d, {1, 0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 9, 0];
nrBAW[9, d, {0, 1, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 36, 0];
nrBAW[9, d, {2, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 3), 72, 0];
nrBAW[9, d, {0, 0, 1, 0, 0, 1}] = If[(d ≥ 3), 84, 0];
nrBAW[9, d, {1, 1, 0, 0, 0, 1}] = If[(d ≥ 3), 252, 0];
nrBAW[9, d, {3, 0, 0, 0, 0, 1}] = If[(d ≥ 4), 504, 0];
nrBAW[9, d, {0, 0, 0, 1, 1}] = If[(d ≥ 2), 126, 0];
nrBAW[9, d, {1, 0, 1, 0, 1}] = If[(d ≥ 3), 504, 0];
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nrBAW[9, d, {0, 2, 0, 0, 1}] = If[(d ≥ 3), 756, 0];
nrBAW[9, d, {2, 1, 0, 0, 1}] = If[(d ≥ 4), 1512, 0];
nrBAW[9, d, {4, 0, 0, 0, 1}] = If[(d ≥ 5), 3024, 0];
nrBAW[9, d, {1, 0, 0, 2}] = If[(d ≥ 3), 630, 0];
nrBAW[9, d, {0, 1, 1, 1}] = If[(d ≥ 3), 1260, 0];
nrBAW[9, d, {2, 0, 1, 1}] = If[(d ≥ 4), 2520, 0];
nrBAW[9, d, {1, 2, 0, 1}] = If[(d ≥ 4), 3780, 0];
nrBAW[9, d, {3, 1, 0, 1}] = If[(d ≥ 5), 7560, 0];
nrBAW[9, d, {5, 0, 0, 1}] = If[(d ≥ 6), 15 120, 0];
nrBAW[9, d, {0, 0, 3}] = If[(d ≥ 3), 1680, 0];
nrBAW[9, d, {1, 1, 2}] = If[(d ≥ 4), 5040, 0];
nrBAW[9, d, {3, 0, 2}] = If[(d ≥ 5), 10 080, 0];
nrBAW[9, d, {0, 3, 1}] = If[(d ≥ 4), 7560, 0];
nrBAW[9, d, {2, 2, 1}] = If[(d ≥ 5), 15 120, 0];
nrBAW[9, d, {4, 1, 1}] = If[(d ≥ 6), 30 240, 0];
nrBAW[9, d, {6, 0, 1}] = If[(d ≥ 7), 60 480, 0];
nrBAW[9, d, {1, 4}] = If[(d ≥ 5), 22 680, 0];
nrBAW[9, d, {3, 3}] = If[(d ≥ 6), 45 360, 0];
nrBAW[9, d, {5, 2}] = If[(d ≥ 7), 90 720, 0];
nrBAW[9, d, {7, 1}] = If[(d ≥ 8), 181 440, 0];;
nrBAW[9, d, {9}] = If[(d ≥ 9), 362 880, 0];

(*n=10,SAW*)

nrSAW[10, d, {8}] = If[(d ≥ 8), 9 031 680 + 2 903 040 (d - 8), 0];
nrSAW[10, d, {6, 1}] = If[(d ≥ 7), 3 669 120 + 1 451 520 (d - 7), 0];
nrSAW[10, d, {6}] = If[(d ≥ 6), 5 342 400 + 5 620 320 (d - 6) + 1 149 120 (d - 6) (d - 7),
   0];
nrSAW[10, d, {5, 0, 1}] = If[(d ≥ 6), 991 200 + 483 840 (d - 6), 0];
nrSAW[10, d, {4, 2}] = If[(d ≥ 6), 1 411 200 + 725 760 (d - 6), 0];
nrSAW[10, d, {4, 1}] = If[(d ≥ 5), 1 442 040 + 2 144 880 (d - 5) + 574 560 (d - 5) (d - 6),
   0];
nrSAW[10, d, {4, 0, 0, 1}] = If[(d ≥ 5), 194 880 + 120 960 (d - 5), 0];
nrSAW[10, d, {3, 1, 1}] = If[(d ≥ 5), 354 480 + 241 920 (d - 5), 0];
nrSAW[10, d, {2, 3}] = If[(d ≥ 5), 493 920 + 362 880 (d - 5), 0];
nrSAW[10, d, {4}] =
  If[(d ≥ 4), 508 608 + 1 990 944 (d - 4) + 1 585 920 (d - 4) (d - 5) +
    297 600 (d - 4) (d - 5) (d - 6), 0];
nrSAW[10, d, {3, 0, 1}] = If[(d ≥ 4), 236 760 + 533 520 (d - 4) + 191 520 (d - 4) (d - 5), 0];
nrSAW[10, d, {2, 2}] = If[(d ≥ 4), 297 960 + 739 800 (d - 4) + 287 280 (d - 4) (d - 5), 0];
nrSAW[10, d, {3, 0, 0, 0, 1}] = If[(d ≥ 4), 28 896 + 24 192 (d - 4), 0];
nrSAW[10, d, {2, 1, 0, 1}] = If[(d ≥ 4), 62 160 + 60 480 (d - 4), 0];
nrSAW[10, d, {2, 0, 2}] = If[(d ≥ 4), 79 520 + 80 640 (d - 4), 0];
nrSAW[10, d, {1, 2, 1}] = If[(d ≥ 4), 106 680 + 120 960 (d - 4), 0];
nrSAW[10, d, {0, 4}] = If[(d ≥ 4), 141 120 + 181 440 (d - 4), 0];
nrSAW[10, d, {2, 1}] =
  If[(d ≥ 3), 41 524 + 392 436 (d - 3) + 538 080 (d - 3) (d - 4) + 148 800 (d - 3) (d - 4) (d - 5),
   0];
nrSAW[10, d, {2, 0, 0, 1}] = If[(d ≥ 3), 22 104 + 92 052 (d - 3) + 47 880 (d - 3) (d - 4), 0];
nrSAW[10, d, {1, 1, 1}] = If[(d ≥ 3), 29 440 + 155 880 (d - 3) + 95 760 (d - 3) (d - 4), 0];
nrSAW[10, d, {0, 3}] = If[(d ≥ 3), 33 210 + 203 580 (d - 3) + 143 640 (d - 3) (d - 4), 0];
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nrSAW[10, d, {2, 0, 0, 0, 0, 1}] = If[(d ≥ 3), 3184 + 4032 (d - 3), 0];
nrSAW[10, d, {1, 1, 0, 0, 1}] = If[(d ≥ 3), 7392 + 12 096 (d - 3), 0];
nrSAW[10, d, {1, 0, 1, 1}] = If[(d ≥ 3), 11 060 + 20 160 (d - 3), 0];
nrSAW[10, d, {0, 2, 0, 1}] = If[(d ≥ 3), 13 440 + 30 240 (d - 3), 0];
nrSAW[10, d, {0, 1, 2}] = If[(d ≥ 3), 16 240 + 40 320 (d - 3), 0];
nrSAW[10, d, {2}] =
  If[(d ≥ 2), 396 + 38 580 (d - 2) + 223 560 (d - 2) (d - 3) + 236 928 (d - 2) (d - 3) (d - 4) +
    54 336 (d - 2) (d - 3) (d - 4) (d - 5), 0];
nrSAW[10, d, {1, 0, 1}] =
  If[(d ≥ 2), 598 + 35 344 (d - 2) + 110 360 (d - 2) (d - 3) + 49 600 (d - 2) (d - 3) (d - 4), 0];
nrSAW[10, d, {0, 2}] =
  If[(d ≥ 2), 568 + 38 700 (d - 2) + 141 600 (d - 2) (d - 3) + 74 400 (d - 2) (d - 3) (d - 4), 0];
nrSAW[10, d, {1, 0, 0, 0, 1}] = If[(d ≥ 2), 636 + 10 548 (d - 2) + 9576 (d - 2) (d - 3), 0];
nrSAW[10, d, {0, 1, 0, 1}] = If[(d ≥ 2), 661 + 18 306 (d - 2) + 23 940 (d - 2) (d - 3), 0];
nrSAW[10, d, {0, 0, 2}] = If[(d ≥ 2), 660 + 21 720 (d - 2) + 31 920 (d - 2) (d - 3), 0];
nrSAW[10, d, {1, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 226 + 576 (d - 2), 0];
nrSAW[10, d, {0, 1, 0, 0, 0, 1}] = If[(d ≥ 2), 416 + 2016 (d - 2), 0];
nrSAW[10, d, {0, 0, 1, 0, 1}] = If[(d ≥ 2), 532 + 4032 (d - 2), 0];
nrSAW[10, d, {0, 0, 0, 2}] = If[(d ≥ 2), 560 + 5040 (d - 2), 0];
nrSAW[10, d, {0, 1}] =
  If[(d ≥ 1), 468 (d - 1) + 19 904 (d - 1) (d - 2) + 59 392 (d - 1) (d - 2) (d - 3) +
    27 168 (d - 1) (d - 2) (d - 3) (d - 4), 0];
nrSAW[10, d, {0, 0, 0, 1}] =
  If[(d ≥ 1), 654 (d - 1) + 11 936 (d - 1) (d - 2) + 12 400 (d - 1) (d - 2) (d - 3), 0];
nrSAW[10, d, {0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 486 (d - 1) + 1596 (d - 1) (d - 2), 0];
nrSAW[10, d, {0, 0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 72 (d - 1), 0];
nrBAW[10, d, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrSAW[10, d, {1, 0, 0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 10, 0];
nrSAW[10, d, {0, 1, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 45, 0];
nrSAW[10, d, {2, 0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 3), 90, 0];
nrSAW[10, d, {0, 0, 1, 0, 0, 0, 1}] = If[(d ≥ 2), 120, 0];
nrSAW[10, d, {1, 1, 0, 0, 0, 0, 1}] = If[(d ≥ 3), 360, 0];
nrSAW[10, d, {3, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 4), 720, 0];
nrSAW[10, d, {0, 0, 0, 1, 0, 1}] = If[(d ≥ 2), 210, 0];
nrSAW[10, d, {1, 0, 1, 0, 0, 1}] = If[(d ≥ 3), 840, 0];
nrSAW[10, d, {0, 2, 0, 0, 0, 1}] = If[(d ≥ 3), 1260, 0];
nrSAW[10, d, {2, 1, 0, 0, 0, 1}] = If[(d ≥ 4), 2520, 0];
nrSAW[10, d, {4, 0, 0, 0, 0, 1}] = If[(d ≥ 5), 5040, 0];
nrSAW[10, d, {0, 0, 0, 0, 2}] = If[(d ≥ 2), 252, 0];
nrSAW[10, d, {1, 0, 0, 1, 1}] = If[(d ≥ 3), 1260, 0];
nrSAW[10, d, {0, 1, 1, 0, 1}] = If[(d ≥ 3), 2520, 0];
nrSAW[10, d, {2, 0, 1, 0, 1}] = If[(d ≥ 4), 5040, 0];
nrSAW[10, d, {1, 2, 0, 0, 1}] = If[(d ≥ 4), 7560, 0];
nrSAW[10, d, {3, 1, 0, 0, 1}] = If[(d ≥ 5), 15 120, 0];
nrSAW[10, d, {5, 0, 0, 0, 1}] = If[(d ≥ 6), 30 240, 0];
nrSAW[10, d, {0, 1, 0, 2}] = If[(d ≥ 3), 3150, 0];
nrSAW[10, d, {2, 0, 0, 2}] = If[(d ≥ 4), 6300, 0];
nrSAW[10, d, {0, 0, 2, 1}] = If[(d ≥ 3), 4200, 0];
nrSAW[10, d, {1, 1, 1, 1}] = If[(d ≥ 4), 12 600, 0];
nrSAW[10, d, {3, 0, 1, 1}] = If[(d ≥ 5), 25 200, 0];
nrSAW[10, d, {0, 3, 0, 1}] = If[(d ≥ 4), 18 900, 0];
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nrSAW[10, d, {2, 2, 0, 1}] = If[(d ≥ 5), 37 800, 0];
nrSAW[10, d, {4, 1, 0, 1}] = If[(d ≥ 6), 75 600, 0];
nrSAW[10, d, {6, 0, 0, 1}] = If[(d ≥ 7), 151 200, 0];
nrSAW[10, d, {1, 0, 3}] = If[(d ≥ 4), 16 800, 0];
nrSAW[10, d, {0, 2, 2}] = If[(d ≥ 4), 25 200, 0];
nrSAW[10, d, {2, 1, 2}] = If[(d ≥ 5), 50 400, 0];
nrSAW[10, d, {4, 0, 2}] = If[(d ≥ 6), 100 800, 0];
nrSAW[10, d, {1, 3, 1}] = If[(d ≥ 5), 75 600, 0];
nrSAW[10, d, {3, 2, 1}] = If[(d ≥ 6), 151 200, 0];
nrSAW[10, d, {5, 1, 1}] = If[(d ≥ 7), 302 400, 0];
nrSAW[10, d, {7, 0, 1}] = If[(d ≥ 8), 604 800, 0];
nrSAW[10, d, {0, 5}] = If[(d ≥ 5), 113 400, 0];
nrSAW[10, d, {2, 4}] = If[(d ≥ 6), 226 800, 0];
nrSAW[10, d, {4, 3}] = If[(d ≥ 7), 453 600, 0];
nrSAW[10, d, {6, 2}] = If[(d ≥ 8), 907 200, 0];
nrSAW[10, d, {8, 1}] = If[(d ≥ 9), 1 814 400, 0];
nrSAW[10, d, {10}] = If[(d ≥ 10), 3 628 800, 0];
(*n=10,BAW*)
nrBAW[10, d, {8}] = If[(d ≥ 8), 9 031 680 + 2 903 040 (d - 8), 0];
nrBAW[10, d, {6, 1}] = If[(d ≥ 7), 3 669 120 + 1 451 520 (d - 7), 0];
nrBAW[10, d, {6}] = If[(d ≥ 6), 5 781 600 + 5 827 680 (d - 6) + 1 169 280 (d - 6) (d - 7),
   0];
nrBAW[10, d, {5, 0, 1}] = If[(d ≥ 6), 991 200 + 483 840 (d - 6), 0];
nrBAW[10, d, {4, 2}] = If[(d ≥ 6), 1 411 200 + 725 760 (d - 6), 0];
nrBAW[10, d, {4, 1}] = If[(d ≥ 5), 1 577 880 + 2 228 400 (d - 5) + 584 640 (d - 5) (d - 6),
   0];
nrBAW[10, d, {4, 0, 0, 1}] = If[(d ≥ 5), 194 880 + 120 960 (d - 5), 0];
nrBAW[10, d, {3, 1, 1}] = If[(d ≥ 5), 354 480 + 241 920 (d - 5), 0];
nrBAW[10, d, {2, 3}] = If[(d ≥ 5), 493 920 + 362 880 (d - 5), 0];
nrBAW[10, d, {4}] =
  If[(d ≥ 4), 683 904 + 2 377 728 (d - 4) + 1 753 632 (d - 4) (d - 5) +
    314 880 (d - 4) (d - 5) (d - 6), 0];
nrBAW[10, d, {3, 0, 1}] = If[(d ≥ 4), 260 760 + 554 640 (d - 4) + 194 880 (d - 4) (d - 5), 0];
nrBAW[10, d, {2, 2}] = If[(d ≥ 4), 334 080 + 771 480 (d - 4) + 292 320 (d - 4) (d - 5), 0];
nrBAW[10, d, {3, 0, 0, 0, 1}] = If[(d ≥ 4), 28 896 + 24 192 (d - 4), 0];
nrBAW[10, d, {2, 1, 0, 1}] = If[(d ≥ 4), 62 160 + 60 480 (d - 4), 0];
nrBAW[10, d, {2, 0, 2}] = If[(d ≥ 4), 79 520 + 80 640 (d - 4), 0];
nrBAW[10, d, {1, 2, 1}] = If[(d ≥ 4), 106 680 + 120 960 (d - 4), 0];
nrBAW[10, d, {0, 4}] = If[(d ≥ 4), 141 120 + 181 440 (d - 4), 0];
nrBAW[10, d, {2, 1}] =
  If[(d ≥ 3), 61 380 + 484 164 (d - 3) + 599 376 (d - 3) (d - 4) + 157 440 (d - 3) (d - 4) (d - 5),
   0];
nrBAW[10, d, {2, 0, 0, 1}] = If[(d ≥ 3), 24 444 + 95 652 (d - 3) + 48 720 (d - 3) (d - 4), 0];
nrBAW[10, d, {1, 1, 1}] = If[(d ≥ 3), 34 200 + 163 080 (d - 3) + 97 440 (d - 3) (d - 4), 0];
nrBAW[10, d, {0, 3}] = If[(d ≥ 3), 40 410 + 214 380 (d - 3) + 146 160 (d - 3) (d - 4), 0];
nrBAW[10, d, {2, 0, 0, 0, 0, 1}] = If[(d ≥ 3), 3184 + 4032 (d - 3), 0];
nrBAW[10, d, {1, 1, 0, 0, 1}] = If[(d ≥ 3), 7392 + 12 096 (d - 3), 0];
nrBAW[10, d, {1, 0, 1, 1}] = If[(d ≥ 3), 11 060 + 20 160 (d - 3), 0];
nrBAW[10, d, {0, 2, 0, 1}] = If[(d ≥ 3), 13 440 + 30 240 (d - 3), 0];
nrBAW[10, d, {0, 1, 2}] = If[(d ≥ 3), 16 240 + 40 320 (d - 3), 0];
nrBAW[10, d, {2}] =
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  If[(d ≥ 2), 1052 + 74 060 (d - 2) + 339 144 (d - 2) (d - 3) + 305 520 (d - 2) (d - 3) (d - 4) +
    63 360 (d - 2) (d - 3) (d - 4) (d - 5), 0];
nrBAW[10, d, {1, 0, 1}] =
  If[(d ≥ 2), 1070 + 45 408 (d - 2) + 123 552 (d - 2) (d - 3) + 52 480 (d - 2) (d - 3) (d - 4), 0];
nrBAW[10, d, {0, 2}] =
  If[(d ≥ 2), 1048 + 52 932 (d - 2) + 160 968 (d - 2) (d - 3) + 78 720 (d - 2) (d - 3) (d - 4), 0];
nrBAW[10, d, {1, 0, 0, 0, 1}] = If[(d ≥ 2), 704 + 10 932 (d - 2) + 9744 (d - 2) (d - 3), 0];
nrBAW[10, d, {0, 1, 0, 1}] = If[(d ≥ 2), 861 + 19 266 (d - 2) + 24 360 (d - 2) (d - 3), 0];
nrBAW[10, d, {0, 0, 2}] = If[(d ≥ 2), 940 + 23 000 (d - 2) + 32 480 (d - 2) (d - 3), 0];
nrBAW[10, d, {1, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 226 + 576 (d - 2), 0];
nrBAW[10, d, {0, 1, 0, 0, 0, 1}] = If[(d ≥ 2), 416 + 2016 (d - 2), 0];
nrBAW[10, d, {0, 0, 1, 0, 1}] = If[(d ≥ 2), 532 + 4032 (d - 2), 0];
nrBAW[10, d, {0, 0, 0, 2}] = If[(d ≥ 2), 560 + 5040 (d - 2), 0];
nrBAW[10, d, {0, 1}] =
  If[(d ≥ 1), 1042 (d - 1) + 33 076 (d - 1) (d - 2) + 78 520 (d - 1) (d - 2) (d - 3) +
    31 680 (d - 1) (d - 2) (d - 3) (d - 4), 0];
nrBAW[10, d, {0, 0, 0, 1}] =
  If[(d ≥ 1), 954 (d - 1) + 13 452 (d - 1) (d - 2) + 13 120 (d - 1) (d - 2) (d - 3), 0];
nrBAW[10, d, {0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 494 (d - 1) + 1624 (d - 1) (d - 2), 0];
nrBAW[10, d, {0, 0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 72 (d - 1), 0];
nrBAW[10, d, {0}] =
  If[(d ≥ 0), 440 d (d - 1) + 11 680 d (d - 1) (d - 2) + 24 720 d (d - 1) (d - 2) (d - 3) +
    9216 d (d - 1) (d - 2) (d - 3) (d - 4), 0];
nrBAW[10, d, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 1), 1, 0];
nrBAW[10, d, {1, 0, 0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 10, 0];
nrBAW[10, d, {0, 1, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 2), 45, 0];
nrBAW[10, d, {2, 0, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 3), 90, 0];
nrBAW[10, d, {0, 0, 1, 0, 0, 0, 1}] = If[(d ≥ 2), 120, 0];
nrBAW[10, d, {1, 1, 0, 0, 0, 0, 1}] = If[(d ≥ 3), 360, 0];
nrBAW[10, d, {3, 0, 0, 0, 0, 0, 1}] = If[(d ≥ 4), 720, 0];
nrBAW[10, d, {0, 0, 0, 1, 0, 1}] = If[(d ≥ 2), 210, 0];
nrBAW[10, d, {1, 0, 1, 0, 0, 1}] = If[(d ≥ 3), 840, 0];
nrBAW[10, d, {0, 2, 0, 0, 0, 1}] = If[(d ≥ 3), 1260, 0];
nrBAW[10, d, {2, 1, 0, 0, 0, 1}] = If[(d ≥ 4), 2520, 0];
nrBAW[10, d, {4, 0, 0, 0, 0, 1}] = If[(d ≥ 5), 5040, 0];
nrBAW[10, d, {0, 0, 0, 0, 2}] = If[(d ≥ 2), 252, 0];
nrBAW[10, d, {1, 0, 0, 1, 1}] = If[(d ≥ 3), 1260, 0];
nrBAW[10, d, {0, 1, 1, 0, 1}] = If[(d ≥ 3), 2520, 0];
nrBAW[10, d, {2, 0, 1, 0, 1}] = If[(d ≥ 4), 5040, 0];
nrBAW[10, d, {1, 2, 0, 0, 1}] = If[(d ≥ 4), 7560, 0];
nrBAW[10, d, {3, 1, 0, 0, 1}] = If[(d ≥ 5), 15 120, 0];
nrBAW[10, d, {5, 0, 0, 0, 1}] = If[(d ≥ 6), 30 240, 0];
nrBAW[10, d, {0, 1, 0, 2}] = If[(d ≥ 3), 3150, 0];
nrBAW[10, d, {2, 0, 0, 2}] = If[(d ≥ 4), 6300, 0];
nrBAW[10, d, {0, 0, 2, 1}] = If[(d ≥ 3), 4200, 0];
nrBAW[10, d, {1, 1, 1, 1}] = If[(d ≥ 4), 12 600, 0];
nrBAW[10, d, {3, 0, 1, 1}] = If[(d ≥ 5), 25 200, 0];
nrBAW[10, d, {0, 3, 0, 1}] = If[(d ≥ 4), 18 900, 0];
nrBAW[10, d, {2, 2, 0, 1}] = If[(d ≥ 5), 37 800, 0];
nrBAW[10, d, {4, 1, 0, 1}] = If[(d ≥ 6), 75 600, 0];
nrBAW[10, d, {6, 0, 0, 1}] = If[(d ≥ 7), 151 200, 0];
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nrBAW[10, d, {1, 0, 3}] = If[(d ≥ 4), 16 800, 0];
nrBAW[10, d, {0, 2, 2}] = If[(d ≥ 4), 25 200, 0];
nrBAW[10, d, {2, 1, 2}] = If[(d ≥ 5), 50 400, 0];
nrBAW[10, d, {4, 0, 2}] = If[(d ≥ 6), 100 800, 0];
nrBAW[10, d, {1, 3, 1}] = If[(d ≥ 5), 75 600, 0];
nrBAW[10, d, {3, 2, 1}] = If[(d ≥ 6), 151 200, 0];
nrBAW[10, d, {5, 1, 1}] = If[(d ≥ 7), 302 400, 0];
nrBAW[10, d, {7, 0, 1}] = If[(d ≥ 8), 604 800, 0];
nrBAW[10, d, {0, 5}] = If[(d ≥ 5), 113 400, 0];
nrBAW[10, d, {2, 4}] = If[(d ≥ 6), 226 800, 0];
nrBAW[10, d, {4, 3}] = If[(d ≥ 7), 453 600, 0];
nrBAW[10, d, {6, 2}] = If[(d ≥ 8), 907 200, 0];
nrBAW[10, d, {8, 1}] = If[(d ≥ 9), 1 814 400, 0];
nrBAW[10, d, {10}] = If[(d ≥ 10), 3 628 800, 0];
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