NoBLE for lattice animals

Implementation of the computer-assisted proof of the NoBLE by Robert Fitzner and Remco van der Hofstad.

Robert Fitzner
Eindhoven University of Technology

Abstract

This document is the second part of the computer-assisted proof of the non-backtracking lace expansion (NoBLE). The NoBLE was created by Remco van der Hofstad and the author to prove mean-field behavior for nearest-neighbor self-avoiding walks, lattice trees (LT), lattice animals (LA) and percolation. In this file the computations for lattice animals are performed. The technique is explained in the paper "Generalized approach to the non-backtracking lace expansion"(I), and the analysis that we implement here are obtained in "NoBLE lattice animals and trees"(II). All references in this file are to these two papers, which we refer to by (I) and (II).

This file is accompanied by two other notebooks -SRW.nb- and -General.nb-. In the -SRW.nb- file a number of simple random walk quantities are computed. In -General.nb- the general bounds, derived in (I) are implemented. Before doing computations with this file, the user should first open these files, choose a dimension and once execute all lines of the file.

Then, the user is expected to choose constants Γ_i and c_μ, $c_{x,s}$ in this file. After choosing these quantities, the user should select the menu item Evaluate-> Evaluate Notebook. In a table at the end of this document the results of the computations are shown: it can be seen whether the bootstrap, with the given parameters, and therefore the analysis, was successful.

The computation of the -SRW.nb- and -General.nb- files are independent of the values Γ_i and c, so that we need to execute these files only once, when you start the Mathematica Kernel or when you want to change the dimension under consideration.

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Input

Here the user should choose the constants Γ_i, c_μ and $c_{n,l,S}$ for which we try to perform the bootstrap, explained in (I), using this program/-document. These values are introduced in Section 2.1, see (2.1)-(2.3) and comments below. For the set S used to define f_3, we are restricted to the sets {0} and $\mathbb{Z}^d \setminus \{0\}$. For the following values the bootstrap succeeds in dimension 17:

In[3100]:=
```
(*The parameter choices that work in d=17 for LAs*)

Gamma1 = 109335749/100000000; (*Assumed bound on zg_z*)
Gamma2 = 124459253/100000000; (* (2d-1)/(2d-2) sup_k[1-D̂(k)]Ĝ_z(k)*)
Gamma3 = 1;
cmu = 1.0333094663;
GammaThree[1, 0] = 0.15664574;
(*Assumed bound on sup_{x≠0} Σ_y |y|_2^2 G(y) G(x-y)*)
GammaThree[1, 1] = 411822/10000000;
(*Assumed bound on sup_{x≠0} Σ_y |y|_2^2 G(y) (D*G) (x-y)*)
GammaThree[1, 2] = 192609/10000000; (*Assumed bound on sup_{x≠0} Σ_y |y|_2^2 G(y) (D*D*G) (x-y)*)
GammaThree[1, 3] = 771789/100000000; (*Assumed bound on sup_{x≠0} Σ_y |y|_2^2 G(y) (D^{*3}*G) (x-y)*)
GammaThree[2, 0] = 2532804/10000000;
(*Assumed bound on sup_{x≠0} Σ_y |y|_2^2 G(y) (G*G) (x-y)*)
GammaThree[2, 1] = 11245001/100000000;
(*Assumed bound on sup_{x≠0} Σ_y |y|_2^2 G(y) (D*G*G) (x-y)*)
GammaThree[2, 2] = 183573/4000000;
(*Assumed bound on sup_{x≠0} Σ_y |y|_2^2 G(y) (D*D*G*G) (x-y)*)
GammaThree[2, 3] = 2025117/100000000;
(*Assumed bound on sup_{x≠0} Σ_y |y|_2^2 G(y) (D^{*3}*G*G) (x-y)*)
GammaThreeClosed[1, 6] = 17725/10000000;
(*Assumed bound on Σ_y |y|_2^2 G(y) (D^{*6}*G) (-y)*)
GammaThreeClosed[2, 6] = 32553/6250000; (*Assumed bound on Σ_y |y|_2^2 G(y) (D^{*6}*G*G) (-y)*)
```

Let us explain how we found the above values: The bound $f_i(z_I)$ at z_I, which is independent of the improvement-of-bound arguments, gives a good orientation. Starting from these values we used a semi-automated procedure to find appropriate values for these constants. The basic idea is based on a fixed-point argument, and works as follow: Starting from some reasonable values we conclude bounds. Then, we use these bounds as our new assumed bounds and re-do all computations again starting from theses values. We repeat this procedure until either the values of the concluded bounds start to diverge (in which case the bootstrap argument fails), or until we actually concluded bounds that were smaller than the bounds assumed in this run (in which case the bootstrap argument is successful). This is implemented near the end of this document.

Sandbox

To improve readability of the code, comments are added. These comments are found between the signs (* comments *). We prepare the following input field in case you want to try other values of the constants. For this, you can just remove the commenting (* / *) in the following input field:

In[3103]:=
```
(*Gamma1=1.08177;
Gamma2=1.227;
Gamma3=1;
cmu=1.03105;
GammaThree[1,0]=0.12288;
GammaThree[1,1]=0.0312;
GammaThree[1,2]=0.014311;
GammaThree[1,3]=0.0055;
GammaThree[2,0]=0.1904;
GammaThree[2,1]=0.081;
GammaThree[2,2]=0.0318;
GammaThree[2,3]=0.01342;
GammaThreeClosed[1,6]=0.001183;
GammaThreeClosed[2,6]=0.00323;*)
```

Bound on the two-point function and on repulsive diagrams

In this section we use the bootstrap assumption $f_i(z) < \Gamma_i$ and the computation of -SRW.nb- to conclude bounds on the two-point function and the basic diagrams

Bound on z and g_z

We define the constants for two setting indicated by s: we use s=i for the bound for $z = z_I$, and s=o for bounds for some z satisfying $z \in (z_I, z_c)$: For $z = z_I$, we use

$$z_I = \frac{1}{(2d-1)e} \qquad g_{z_I} \le e + \frac{e-1}{2d-1} \qquad g^i_{z_I} \le 1 + (g_{z_I} - 1)\frac{2d-1}{2d} \le e$$
$$\overline{G}_z(x) \le g^i_{z_I} B_{z_I g^i_{z_I}}(x) \le g^i_{z_I} \frac{2d-2}{2d-1} C(x) \qquad \tilde{G}_z(0) = 1, \tag{1}$$

$$\text{and otherwise } \tilde{G}_z(x) \le B_{z_I g^i_{z_I}}(x) \le \frac{2d-2}{2d-1} C(x). \tag{2}$$

and implement this:

In[3104]:=
```
z[i] = 1/((2 d - 1) Exp[1]);
gj[i] = Exp[1];
g[i] = Exp[1] + (Exp[1] - 1)/(2 d - 1);
gz[i] = g[i] × z[i];
gjz[i] = gj[i] × z[i];
VarGamma2[i] = 1;
VarGamma2b[i] = VarGamma2[i] gz[i]/gjz[i];
```

For $z \in (z_I, z_c)$, we assume that

$$2d\, z\, g^\iota_z < 2d\, g_z\, z < \Gamma_1, \qquad g_z < e\,\Gamma_1 \text{ for all } z \ge \frac{1}{(2d-1)e}, \qquad g^\iota_z < 1 + (g_z - 1)\frac{2d-1}{2d}. \tag{3}$$

In[3111]:=
```
gz[o] = Gamma1/(2 d - 1);
gjz[o] = 1/cmu Gamma1/(2 d - 1);
g[o] = Exp[1] Gamma1;
gj[o] = Min[Exp[1] cmu Gamma1, 1 + (g[o] - 1) (2 d - 1)/(2 d)];
VarGamma2[o] = Gamma2;
VarGamma2b[o] = VarGamma2[o] gz[o]/gjz[o];
```

Lower bound on g_z

For some bounds, we require a bound on $z < z_c$. For this we prove a lower bound on g_z, which we obtain by using that we only consider $z > z_I = \frac{1}{(2\,d-1)\,e}$ and the following basic bounds:

$$\begin{gathered} g_z = \Sigma_{n=0}\, t_n(0)\, z^n \geq \Sigma_{n=0}^{3}\, (\text{lower bound on } t_n(0))\, z_I^n \\ (n=0) \implies t_n(0)\, z_I^n = 1 \\ (n=1) \implies t_1(0)\, z_I^1 = 2\,d * \frac{1}{(2\,d-1)\,e} > \frac{1}{e} \\ (n=1) \implies t_2(0)\, z_I^2 = 3\,d\,(2\,d-1) * \frac{1}{(2\,d-1)^2\,e^2} = \frac{3}{2}\,\frac{d}{d-\frac{1}{2}}\,\frac{1}{e^2} > \frac{1.5}{e^2} \end{gathered} \tag{4}$$

In[3117]:=
```
gLower = 1 + 1/Exp[1] + 1.5/Exp[2];
"Which is " <> TextString[NumberForm[N[gLower/g[o] * 100], 5]] <>
 "% of the upper bounds."
z[o] = gz[o]/gLower;
```

Out[3118]= Which is 52.855% of the upper bounds.

We could use more terms than just the first three and improve our bounds further to up to 70% of the upper bound. We found however that this will not reduce the dimension for which we obtain our main results.

Bounds on two-point functions

Here, we compute bounds on the two-point functions for some x, as explained in Section 5.3.2. of (I). These bounds are obtained by extracting short, explicit contributions and by bounding the longer contributions using f_2, see Section 5.3 of (I). To bound the short explicit contributions we use the values $c_j(x)$ =nrSAW[j,d,x] provided in the model-independent SRW-integral notebook, these are provided for $j\leq$ *ComputedSteps*. The value of *ComputedSteps* is given in the -SRW.nb- file. The extraction of short contributions creates a better bound than just applying f_2. The reason is that f_1 gives a sharper bound than f_2. Thus, we rely on

$$\tilde{G}_{m,z}(x) \leq \sum_{j=m}^{M-1} (g_z^{\iota}\, z)^j\, c_j(x) + \tilde{G}_{M,z}(x). \tag{5}$$

For lattice animals we have in total defined three different two-point functions, namely $G_{m,z}, \overline{G}_{m,z}, \tilde{G}_{m,z}$. The function $G_{m,z}, \overline{G}_{m,z}$ can be bounded directly using f_2. For $\tilde{G}_{m,z}$ we use the additional bound that

$$\tilde{G}_{m,z}(x) \leq 2\,d\,z\,(D * \overline{G}_{m-1,z})\,(x) = 2\,d\,z\,g_z\,(D * G_{m-1,z})\,(x) \leq 2\,d\,g_z\,(2\,d\,z\,g_z^{\iota})^{m-1}\,(D^m * G_z)\,(x) \leq 2\,d\,g_z\,(2\,d\,z\,g_z^{\iota})^{m-1}\,\overline{\Gamma}_2\,K_{1,m}(x). \tag{6}$$

for even m. We use the following variables, that are required to be even:

```
In[3120]:= (*Number of steps we will extract,
  we have to choose M=Rsteps to apply the bound.*)
  If[EvenQ[ComputedSteps],
   Explicit = ComputedSteps;
   (*the length of the polygon that we extract. It is supposed to be even*)
   RSteps = ComputedSteps + 2;,
   (*The length of the shortest loop in the remainder terms*)
   Explicit = ComputedSteps - 1;
   RSteps = ComputedSteps + 1;]
```

Bound on $G_{m,z}(e_1)$

We use $G_{m,z}(e_1) = (D * G_{m,z})(0)$ to compute

```
In[3121]:= Do[
   Do[
    Bound[tG, {1}, m, s] = Sum[gjz[s]^j nrSAW[j, d, {1}], {j, m, Explicit}] +
       2 d gz[s] (2 d gjz[s])^(RSteps-2) VarGamma2[s] × Ivalue[1, RSteps, {0}];
     , {m, 1, 5}]
   , {s, {i, o}}]
```

Bound on $G_{m,z}(e_1 + e_2)$

We use $G_{m,z}(e_1 + e_2) \le \frac{d}{d-1} \left(D^2 * G_{m,z}\right)(0)$ to bound

```
In[3122]:= Do[
   Do[
    Bound[tG, {2}, 2 m, s] = Sum[gjz[s]^j nrSAW[j, d, {2}], {j, 2 m, Explicit}] +
       d/(d - 1) 2 d gz[s] (2 d gjz[s])^(RSteps-1) VarGamma2[s] × Ivalue[1, RSteps + 2, {0}];
     , {m, 1, ComputedSteps / 2}]
   , {s, {i, o}}]
```

We are also interested in bounds on the constrained connection that does not use the lattice point e_ι. We obtain a bound on this by removing some of the explicit paths using the point e_1.

```
In[3123]:= Do[
   Bound[tG, ikNotUsingi, 6, s] =
    gjz[s]^6 (nrSAW[6, d, {2}] - 9 (2 d - 3)^2 - 2 × (2 d - 4) - 2 × (2 d - 3)) +
     Bound[tG, {2}, 8, o];
   Bound[tG, ikNotUsingi, 4, s] =
    gjz[s]^4 (nrSAW[4, d, {2}] - 2 (d - 2)) + Bound[tG, ikNotUsingi, 6, s];
   Bound[tG, ikNotUsingi, 2, s] = gjz[s]^2 + Bound[tG, ikNotUsingi, 4, s];
   , {s, {i, o}}]
```

Bound on $G_{m,z}(2\,e_1)$

Now, we compute bounds on $G_{m,z}(2\,e_1)$ and $G^1_{m,z}(2\,e_1)$. We obtain the latter by subtracting a number of explicit contributions.

In[3124]:=
```
Do[
 Do[
  Bound[tG, {0, 1}, 2 m, s] = Sum[gjz[s]^j nrSAW[j, d, {0, 1}], {j, 2 m, Explicit}] +
     2 d gz[s] (2 d gjz[s])^(RSteps-1) VarGamma2[s] × Ivalue[1, RSteps, {0}];
  , {m, 1, ComputedSteps / 2}];
 Bound[tG, twoiNotusingi, 6, s] =
  gjz[s]^6 (nrSAW[6, d, {0, 1}] - 36 (d - 2)^2) + Bound[tG, {0, 1}, 8, o];
 Bound[tG, twoiNotusingi, 4, s] = gjz[s]^4 2 (d - 1) + Bound[tG, twoiNotusingi, 6, s];
 Bound[tG, twoiNotusingi, 2, s] = Bound[tG, twoiNotusingi, 4, s];
 , {s, {i, o}}]
```

Bound on $\sup_{x\neq 0} G_{m,z}(x)$

To compute the supremum of the two-point function we use that $c_n(x)$=nrSAW[n,d,x] for $n\leq 10$ has its maximal value at $x=e_1$ or $x=e_1+e_2$.

In[3125]:=
```
maxpossible = Max[Explicit, MaxNumberOfSteps / 2];
Do[
 Bound[tG, max, maxpossible, s] =
  (2 d gjz[s])^maxpossible VarGamma2[s] × K[1, maxpossible, {1}];
 Do[
  m = maxpossible - 2 t;
  Bound[tG, max, m, s] = Max[nrSAW[m, d, {2}] gjz[s]^m, nrSAW[m + 1, d, {1}] gjz[s]^(m+1)] +
    Bound[tG, max, m + 2, s];
  , {t, 1, maxpossible / 2 - 1}];
 Bound[tG, max, 5, s] = Max[Bound[tG, max, 6, s], Bound[tG, {1}, 5, s]];
 Bound[tG, max, 3, s] = Max[Bound[tG, max, 4, s], Bound[tG, {1}, 3, s]];
 Bound[tG, max, 1, s] = Max[Bound[tG, max, 2, s], Bound[tG, {1}, 1, s]];
 Bound[tG, max, 0, s] = Bound[tG, max, 1, s];
 , {s, {i, o}}]
Clear[t, m, explicit, maxpossible]
```

Bound on $\frac{g_z}{g_z^\iota}$ and $\frac{g_z^\iota}{g_z}$

We require an upper on $\frac{\overline{\mu}_z}{\mu_z}$, which we conclude from

$$\frac{\overline{\mu}_z}{\mu_z}=\frac{z g_z}{z g_z^\iota}=\frac{g_z}{g_z^\iota}=1+\frac{g_z-g_z^\iota}{g_z^\iota}=1+\frac{1}{g_z^\iota}\,\overline{G}_z(e_1)\leq\ 1+\tilde{G}_z(e_1).$$

The expression Bound[tG,{1},3,s] gives an upper bound on $\tilde{G}_{3,z}(e_1)$. For the lower bound we use that $\overline{G}_z(e_1)>0$, so that $\frac{\overline{\mu}_z}{\mu_z}>1$ and $\frac{\mu_z}{\overline{\mu}_z}<1$.

In[3128]:=
```
Do[
 mubOverMu[s] = 1 + Bound[tG, {1}, 1, s];
 muOverMub[s] = 1;
 , {s, {i, o}}]
```

Repulsive diagrams

Now, we bound repulsive diagrams, defined in Definition 4.7 of (II) as described in Section 5.3.2 of (I). In (I) we see that the resulting bounds do no depend on the individual lengths of the pieces m_1, m_2, ... , but only of the sum of the known lengths $\sum_i m_i$. So we refer to each diagram by the number of minimal steps $\sum_i m_i$ and the number of two-point functions without fixed length ($G_{m,z}(x)$ instead of $G_{\underline{m},z}(x)$).

Further, we assume that only two-point functions $\tilde{G}_{m,z}$ are involved.

Closed diagrams

In[3129]:=
```
Do[
  Bound[Loop, 4, s] = 2 d gjz[s] × Bound[tG, {1}, 3, s];
  Do[
   Bound[Bubble, m, s] =
     Sum[(j + 1 - m) 2 d nrBAW[j - 1, d, {1}] gjz[s]^j, {j, m, Explicit}] +
      (RSteps - m) (2 d gjz[s])^RSteps VarGamma2b[s] × Ivalue[1, RSteps, {0}] +
      (2 d gjz[s])^RSteps VarGamma2b[s]^2 Ivalue[2, RSteps, {0}]; (* (5.40) of I*)
   , {m, 2, 8}];
  Bound[Double, 2, s] = Bound[Loop, 4, s] + Bound[Bubble, 4, s]/2;
  Bound[Double, 4, s] = Bound[Bubble, 4, s]/2;
  Do[
   Bound[Triangle, m, s] =
     Sum[((j + 1 - m) (j + 2 - m))/2 2 d nrBAW[j - 1, d, {1}] gjz[s]^j, {j, m, Explicit}] +

      ((RSteps - m) (RSteps - 1 - m))/2 (2 d gjz[s])^RSteps VarGamma2b[s] ×
       Ivalue[1, RSteps, {0}] + (RSteps + 1 - m) (2 d gjz[s])^RSteps VarGamma2b[s]^2
       Ivalue[2, RSteps, {0}] + (2 d gjz[s])^RSteps VarGamma2b[s]^3 Ivalue[3, RSteps , {0}];
   (* (5.41) of I*)
   , {m, 1, 5}];
  Do[
   Bound[Square, m, s] =
     Sum[((j + 1 - m) (j + 2 - m) (j + 3 - m))/6 2 d nrBAW[j - 1, d, {1}] gjz[s]^j,
       {j, m, Explicit}] + 1/6 (RSteps + 1 - m) (RSteps + 2 - m) (RSteps + 3 - m)
       (2 d gjz[s])^RSteps VarGamma2b[s] × Ivalue[1, RSteps, {0}] +
      ((RSteps - m) (RSteps - 1 - m))/2 (2 d gjz[s])^RSteps VarGamma2b[s]^2
       Ivalue[2, RSteps, {0}] + (RSteps - m) (2 d gjz[s])^RSteps VarGamma2b[s]^3
       Ivalue[3, RSteps, {0}] + gz[s]/gjz[s] (2 d gjz[s])^RSteps VarGamma2b[s]^4
       Ivalue[4, RSteps, {0}]; (* (5.42) of I*)
   , {m, 2, 5}];
  , {s, {i, o}}]
```

Open repulsive diagrams

We can bound open repulsive diagrams in the same way as the closed diagrams. Parallel to this, we also produce the bounds without extracting contributions, as it is a priori not clear which bound is better. We use the monotonicity of the SRW-integrals, see Lemma 5.1 of (I): $\sup_{x\neq 0} K_{n,l}(x) = K_{n,l}(e_1)$, to conclude that the supremum of the open diagrams is at a neighbor of the origin.

```
In[3130]:= Do[
  Do[
    Bound[OpenBubblePre, m, s] = Min[gz[s]/gjz[s] (2 d gjz[s])^m VarGamma2[s]^2 K[2, m, {1}],
      Max[Sum[(j + 1 - m) gjz[s]^j nrBAW[j, d, {2}], {j, m, ComputedSteps}],
        Sum[(j + 1 - m) gjz[s]^j nrBAW[j, d, {1}], {j, m, ComputedSteps}]]
       + (ComputedSteps + 1 - m) (2 d gjz[s])^(ComputedSteps+1) VarGamma2b[s] ×
        K[1, ComputedSteps + 1, {1}] +
       (2 d gjz[s])^(ComputedSteps+1) VarGamma2b[s]^2 K[2, ComputedSteps + 1, {1}]];
    Bound[OpenTrianglePre, m, s] =
     Min[(2 d gjz[s])^m VarGamma2b[s]^3 K[3, m, {1}],
      Max[Sum[((j + 1 - m) (j + 2 - m))/2 gjz[s]^j nrBAW[j, d, {2}], {j, m, ComputedSteps}],
        Sum[((j + 1 - m) (j + 2 - m))/2 gjz[s]^j nrBAW[j, d, {1}], {j, m, ComputedSteps}]] +
       1/2 (ComputedSteps + 1 - m) (ComputedSteps + 1 - 1 - m) (2 d gjz[s])^(ComputedSteps+1)
        VarGamma2b[s] × K[1, ComputedSteps + 1, {1}] +
       (ComputedSteps + 1 - m) (2 d gjz[s])^(ComputedSteps+1) VarGamma2b[s]^2
        K[2, ComputedSteps + 1, {1}] +
       (2 d gjz[s])^(ComputedSteps+1) VarGamma2b[s]^3 K[3, ComputedSteps + 1, {1}]];
    Bound[OpenSquarePre, m, s] = Min[(2 d gjz[s])^m VarGamma2b[s]^4 K[4, m, {1}],
      Max[Sum[1/6 (j + 1 - m) (j + 2 - m) (j + 3 - m) gjz[s]^j nrBAW[j, d, {2}],
         {j, m, ComputedSteps}],
        Sum[1/6 (j + 1 - m) (j + 2 - m) (j + 3 - m) gjz[s]^j nrBAW[j, d, {1}],
         {j, m, ComputedSteps}]] +
       1/6 × (10 + 1 - m) × (10 + 2 - m) × (10 + 3 - m) (2 d gjz[s])^(ComputedSteps+1)
        VarGamma2b[s] × K[1, ComputedSteps + 1, {1}] +
       ((10 - m) × (10 - 1 - m))/2 (2 d gjz[s])^(ComputedSteps+1) VarGamma2b[s]^2
        K[2, ComputedSteps + 1, {1}] +
       (10 - m) (2 d gjz[s])^(ComputedSteps+1) VarGamma2b[s]^3 K[3, ComputedSteps + 1, {1}] +
       (2 d gjz[s])^(ComputedSteps+1) VarGamma2b[s]^4 K[4, ComputedSteps + 1, {1}]];
    , {m, 0, 6}];
  , {s, {i, o}}]
```

When trying to find an optimal bound we saw that the following idea improved the bounds slightly, allowing us to prove the bound for one additional dimension. Let us show the idea in the example of a repulsive triangle with some length restrictions:

$$T_{1,0,2}(x) \leq T_{\underline{1},0,2}(x) + T_{2,0,2}(x) \leq T_{\underline{1},0,2}(x) + T_{\underline{2},0,2}(x) + T_{\underline{3},0,2}(x) + T_{4,0,2}(x)$$

In[3131]:=
```
Do[
  m = 6;
  Bound[OpenBubble, m, s] = Bound[OpenBubblePre, m, s];
  Bound[OpenTriangle, m, s] = Bound[OpenTrianglePre, m, s];
  Bound[OpenSquare, m, s] = Bound[OpenSquarePre, m, s];
  Clear[m];
  , {s, {i, o}}];
```

Using this initial step we iteratively consider smaller and smaller diagrams and in each step use the best bound possible:

In[3132]:=
```
Do[
  Do[
    m = 5 - it;
    Bound[OpenBubble, m, s] = Min[Bound[OpenBubblePre, m, s],
      Bound[tG, max, m, s] + Bound[OpenBubble, m + 1, s]];
    Bound[OpenTriangle, m, s] = Min[Bound[OpenTrianglePre, m, s],
      Bound[OpenBubble, m, s] + Bound[OpenTriangle, m + 1, s]];
    Bound[OpenSquare, m, s] = Min[Bound[OpenSquarePre, m, s],
      Bound[OpenTriangle, m, s] + Bound[OpenSquare, m + 1, s]];
    , {it, 0, 5}];
  , {s, {i, o}}];
```

Weighted Diagrams

Here we define the bounds on the weighted diagrams. As explained in Section 5.3.3 of (I) for $z = z_I$ we use a bound that is independent of our analysis. These bounds have been implemented in the accompanying notebook. To explain the notation, we remark that, similarly as for the unweighted diagrams, we refer to the diagrams by their number of two-point functions and number of the fixed steps on the unweighted lines.

In[3133]:=
```
Bound[WeightedBubble, 6, i] = (2 d gjz[i])^6 BoundFThreeInital[d, 1, 6, 1, {{0}}] ;
Bound[WeightedTriangle, 6, i] = (2 d gjz[i])^6 BoundFThreeInital[d, 2, 6, 1, {{0}}] ;

Do[
 Bound[WeightedOpenBubble, t, i] =
  (2 d gjz[i])^t BoundFThreeInital[d, 1, t, 1, {{1}, {2}, {0, 1}}];
 Bound[WeightedOpenTriangle, t, i] =
  (2 d gjz[i])^t BoundFThreeInital[d, 2, t, 1, {{1}, {2}, {0, 1}}];
 , {t, 0, 3}]
```

For $z \in (z_I, z_c)$, we use the bootstrap function f_3 to obtain the bounds

In[3136]:=
```
Bound[WeightedBubble, 6, o] = (2 d gjz[o])^6 GammaThreeClosed[1, 6];
Bound[WeightedTriangle, 6, o] = (2 d gjz[o])^6 GammaThreeClosed[2, 6];

Do[
 Bound[WeightedOpenBubble, t, o] = (2 d gjz[o])^t GammaThree[1, t];
 Bound[WeightedOpenTriangle, t, o] = (2 d gjz[o])^t GammaThree[2, t];
 , {t, 0, 3}]
```

As explained in Section 5.3.3, we drastically improve our bounds on closed, weighted, and repulsive diagrams by extracting explicit contributions, by using their repulsiveness as well as

$$\frac{1}{g_z}\sum_x \|x\|_2^2 \sum_{A,\, x\epsilon A} z^A \quad\le\quad \frac{1}{g_z}\sum_x \|x\|_2^2\, \overline{G}_z(x)\, 2\, d\, z\left(D*\tilde{G}\right)(x) \le \sum_x \|x\|_2^2\, G_z(x)\, 2\, d\, z\left(D*\tilde{G}\right)(x)$$

$$\frac{1}{g_z}\sum_x \|x\|_2^2 \sum_{A,\, x\epsilon A} 1_{d(0,x)>n}\; z^A \quad\le\quad \sum_x \|x\|_2^2\, G_z(x)\, \frac{\left(2\, d\, z\, g_z^t\right)^n}{g_z^t}\left(D^{*n}*\tilde{G}\right)(x)$$

We bound the unweighted connections $\left(G^{*n}*D^{*l}\right)(x)$ using $K_{n,l}$ (x), defined in (3.36) of (I). We compute $K_{n,l}(x)$ for $l \le rem$, see below, and some points close to the origin. For x for which we have not computed $K_{n,l}(x)$ we use a monotonicity argument to bound $K_{n,l}(x)$, in our case $K(2 e_1) > K(x)$ for all relevant x. This monotonicity is implied by Lemma 5.1 of (I).

```
In[3139]:= Do[
    explicit = Min[MaxNumberOfSteps / 2 - 2, ComputedSteps];
    rem = explicit + 1;
    LongContributions[point_] = (2 d gz[s]) (2 d gjz[s])^(rem-1) K[1, rem, point];
    (*  and the points x for which we have not computed K(x). We bound
      their values by K(2e1)>K(x).*)
    point5Remainder = {{1, 1}, {0, 0, 1}, {0, 0, 0, 0, 1}, {1, 0, 0, 1}, {0, 1, 1},
      {2, 0, 1}, {1, 2}, {3, 1}};
    point4Remainder = {{0, 1}, {1, 0, 1}, {0, 0, 0, 1}, {0, 2}, {2, 1}};

    Bound[WeightedBubble, 5, s] =
     Bound[WeightedBubble, 6, s] +
       1
      ------
      gLower
       (gjz[s]^5 nrBAW[5, d, {1}] *
          (Sum[ nrBAW[r, d, {1}] gjz[s]^r, {r, 1, explicit}] + LongContributions[{1}]) +
        3 gjz[s]^5 nrSAW[5, d, {3}] *
          (Sum[ nrBAW[r, d, {3}] gjz[s]^r, {r, 1, explicit}] + LongContributions[{2}]) +
        5 gjz[s]^5 nrSAW[5, d, {5}] *
          (Sum[ nrBAW[r, d, {5}] gjz[s]^r, {r, 1, explicit}] + LongContributions[{2}]) +
        25 gjz[s]^5
          Sum[nrSAW[5, d, v] * (Sum[ nrBAW[r, d, v] gjz[s]^r, {r, 1, explicit}] +
              LongContributions[{0, 1}]), {v, point5Remainder}]);

    Bound[WeightedBubble, 4, s] =
     Bound[WeightedBubble, 5, s] +
       1
      ------
      gLower
       (2 gjz[s]^4 nrBAW[4, d, {2}] *
          (Sum[ nrBAW[r, d, {2}] gjz[s]^r, {r, 1, explicit}] + LongContributions[{2}]) +
        4 gjz[s]^4 nrSAW[4, d, {4}] *
          (Sum[ nrBAW[r, d, {4}] gjz[s]^r, {r, 1, explicit}] + LongContributions[{2}]) +
        10 Sum[gjz[s]^4 nrSAW[4, d, v] *
            (Sum[ nrBAW[r, d, v] gjz[s]^r, {r, 1, explicit}] + LongContributions[{0, 1}]),
          {v, point4Remainder}]);

    Bound[WeightedBubble, 3, s] =
     Bound[WeightedBubble, 4, s] +
```

```
 1
─────── 
gLower
 (gjz[s]^3 nrSAW[3, d, {1}] *
    (Sum[ nrBAW[r, d, {1}] gjz[s]^r, {r, 1, explicit}] + LongContributions[{1}]) +
   5 gjz[s]^3 nrSAW[3, d, {1, 1}] *
    (Sum[ nrBAW[r, d, {1, 1}] gjz[s]^r, {r, 1, explicit}] +
      LongContributions[{0, 1}]) +
   9 gjz[s]^3 nrSAW[3, d, {0, 0, 1}] *
    (Sum[ nrBAW[r, d, {0, 0, 1}] gjz[s]^r, {r, 1, explicit}] +
      LongContributions[{0, 1}]) +
   3 gjz[s]^3 nrSAW[3, d, {3}] *
    (Sum[ nrBAW[r, d, {3}] gjz[s]^r, {r, 1, explicit}] + LongContributions[{2}]));

Bound[WeightedBubble, 2, s] =
 Bound[WeightedBubble, 3, s] +
   1
 ─────── (2 d * 4 * gjz[s]^2 Bound[tG, twoiNotusingi, 4, s] +
 gLower
   2 * 2 * gjz[s]^2 2 d (2 d - 2) Bound[tG, ikNotUsingi, 2, s]);
Bound[WeightedBubble, 1, s] =
 Bound[WeightedBubble, 2, s] + 2 d z[s] × Bound[tG, {1}, 3, s];
Bound[WeightedBubble, 0, s] = Bound[WeightedBubble, 1, s];

Bound[WeightedTriangle, 5, s] =
 Bound[WeightedBubble, 5, s] + Bound[WeightedTriangle, 6, s];
Bound[WeightedTriangle, 4, s] =
 Bound[WeightedBubble, 4, s] + Bound[WeightedTriangle, 5, s];
Bound[WeightedTriangle, 3, s] =
 Bound[WeightedBubble, 3, s] + Bound[WeightedTriangle, 4, s];
Bound[WeightedTriangle, 2, s] =
 Bound[WeightedBubble, 2, s] + Bound[WeightedTriangle, 3, s];
Bound[WeightedTriangle, 1, s] =
 Bound[WeightedBubble, 1, s] + Bound[WeightedTriangle, 2, s];
Bound[WeightedTriangle, 0, s] =
 Bound[WeightedBubble, 0, s] + Bound[WeightedTriangle, 1, s];
(*remove the auxilliary variables from the memory*)
Clear[point4Remainder, point5Remainder, LongContributions];

                                Bound[WeightedBubble, 2, s]
Bound[WeightedDouble, 2, s] = ─────────────────────────────;
                                            2
Bound[WeightedDouble, 1, s] =
 Bound[WeightedDouble, 2, s] + 2 d z[s] × Bound[tG, {1}, 3, s];
, {s, {i, o}}]
```

Building Blocks

Blocks without weight

In the following we implement the bound on the coefficient $A^{m,l}$ defined in Appendix C.1.

```
In[3140]:= Do[
  Bound[A, 0, 0, s] = Bound[Loop, 4, s] + 2 Bound[Bubble, 3, s] + Bound[Triangle, 3, s];
```

```
(*Table 2*)
Bound[A, 0, 1, s] = Bound[Loop, 4, s] + 2 Bound[Bubble, 3, s] + Bound[Triangle, 4, s];
(*Table 3*)
Bound[A, 0, 2, s] = Bound[Bubble, 3, s] + 2 Bound[Triangle, 4, s] + Bound[Square, 5, s];
(*Table 4*)
Bound[A, 0, -1, s] = Bound[Loop, 4, s] (*d0,x=1,w=y*) + Bound[Bubble, 4, s]
  (*d0,x=1,0≠w≠y*) + Bound[Triangle, 3, s] (*d0,x>2*); (*Table 5*)
Bound[A, 0, -2, s] = Bound[Triangle, 4, s] (*d0,x=1→w≠0*) + Bound[Triangle, 4, s]
  (*d0,x≥2,w=0*) + Bound[Square, 5, s] (*d0,x≥2,w≠0*);
(*Table 6*)



Bound[A, -1, 0, s] = Bound[Bubble, 3, s]/(2 d gjz[s]) + Bound[Triangle, 3, s]/(2 d gjz[s]);
Bound[A, -1, -1, s] = Bound[Triangle, 3, s]/(2 d gjz[s]);
Bound[A, -1, -2, s] = Bound[Square, 4, s]/(2 d gjz[s]);
Bound[A, -1, 1, s] = Bound[Triangle, 3, s]/(2 d gjz[s]);
Bound[A, -1, 2, s] = Bound[Square, 4, s]/(2 d gjz[s]);

Bound[A, -2, 0, s] = Bound[OpenBubble, 2, s] + Bound[OpenTriangle, 2, s];
Bound[A, -2, 1, s] = Bound[OpenTriangle, 2, s];
Bound[A, -2, -1, s] = Bound[OpenTriangle, 2, s];
Bound[A, -2, 2, s] = Bound[OpenSquare, 3, s];
Bound[A, -2, -2, s] = Bound[OpenSquare, 3, s];
Do[Do[
  Bound[A, a, b, s] = Bound[A, -a, b, s];
  , {a, {1, 2}}], {b, {-2, -1, 0, 1, 2}}];

Do[
 Bound[Abar, b, 0, s] = Bound[A, b, 0, s];
 Bound[Abar, 0, b, s] = Bound[Abar, b, 0, s]
 , {b, -2, 2}];
Do[
 Bound[Abar, b, 1, s] = Bound[Abar, 0, 1, s]/(2 d gjz[s]);
 Bound[Abar, b, -1, s] = Bound[Abar, 0, -1, s]/(2 d gjz[s]);
 , {b, {-1, 1}}];

Do[Do[
  Bound[Abar, a, b, s] = (2 d gjz[s])^1 VarGamma2[s]^3 K[3, 1, {0}];
```

```
  , {a, {-2, 2}}], {b, {-2, 2}}];

Do[Do[
  Bound[Abar, a, b, s] = (2 d gjz[s])^1 VarGamma2[s]^3 K[3, 2, {1}];
  Bound[Abar, b, a, s] = (2 d gjz[s])^1 VarGamma2[s]^3 K[3, 2, {1}];
  , {a, {-1, 1}}], {b, {-2, 2}}];
, {s, {i, o}}];
```

Blocks with weight

Here we implement the elements of our bounds stated in Appendix C.2.

```
In[3141]:= Do[
  Bound[C, 0, 0, s] = 2 Bound[WeightedBubble, 2, s] + Bound[WeightedTriangle, 2, s];
  Bound[C, 0, -1, s] = Bound[WeightedBubble, 2, s] / (2 d gjz[s]) + Bound[WeightedTriangle, 3, s] / (2 d gjz[s]);
  Bound[C, -1, 0, s] = Bound[C, 0, -1, s];
  Bound[C, 0, 1, s] = Bound[C, 0, -1, s];
  Bound[C, 1, 0, s] = (*d_{0,x}=1,v=w≠ x*)
   (4 * gjz[s] × Bound[tG, twoiNotusingi, 4, s] +
      2 * gjz[s] (2 d - 2) Bound[tG, ikNotUsingi, 2, s]) + (*d_{0,x}=1,v≠w≠ x,
    split weight*) 2 Bound[WeightedBubble, 3, s] / (2 d gjz[s]) + (*d_{0,x}≥ 2,split weight*)
    2 Bound[WeightedTriangle, 3, s] / (2 d gjz[s]);
  Bound[C, -1, -1, s] = (*v=w=y*)
   Max[4 * Bound[tG, twoiNotusingi, 4, s], 2 * Bound[tG, ikNotUsingi, 2, s]] +
    (*v≠w=y or v=w≠y*) 2 Bound[WeightedBubble, 3, s] / (2 d gjz[s])^2 + (*v≠w≠y*)
    Bound[WeightedTriangle, 4, s] / (2 d gjz[s])^2;
  Bound[C, -1, 1, s] = Bound[C, -1, -1, s];
  Bound[C, 1, -1, s] = 2 Bound[WeightedBubble, 3, s] / (2 d gjz[s])^2 +
    4 Bound[WeightedTriangle, 4, s] / (2 d gjz[s])^2;
  Bound[C, 1, 1, s] = Bound[C, 1, -1, s];
  (*the bad bounds*)
  Bound[C, 0, 2, s] = Bound[WeightedOpenTriangle, 0, s];
  Bound[C, 0, -2, s] = Bound[WeightedOpenTriangle, 0, s];
  Do[
   Bound[C, -2, t, s] = Bound[WeightedOpenTriangle, 0, s];
   Bound[C, 2, t, s] = 2 Bound[WeightedOpenTriangle, 0, s];
   , {t, {-2, 0, 2}}];
```

```
Bound[C, -2, 1, s] = Bound[WeightedOpenTriangle, 1, s] / (2 d gjz[s]);
Bound[C, -2, -1, s] = Bound[WeightedOpenTriangle, 1, s] / (2 d gjz[s]);
Bound[C, -1, 2, s] = Bound[WeightedOpenTriangle, 1, s] / (2 d gjz[s]);
Bound[C, -1, -2, s] = Bound[WeightedOpenTriangle, 1, s] / (2 d gjz[s]);


Bound[C, 2, 1, s] = 2 Bound[WeightedOpenTriangle, 1, s] / (2 d gjz[s]);
Bound[C, 2, -1, s] = 2 Bound[WeightedOpenTriangle, 1, s] / (2 d gjz[s]);
Bound[C, 1, 2, s] = 2 Bound[WeightedOpenTriangle, 1, s] / (2 d gjz[s]);
Bound[C, 1, -2, s] = 2 Bound[WeightedOpenTriangle, 1, s] / (2 d gjz[s]);
, {s, {i, o}}]
```

Initial Blocks

Here we define the initial and terminal part of the diagrams as described in Appendix C.3.

```
In[3142]:= Do[
  Bound[Start, 0, s] = (1 + Bound[Double, 2, s]);
  Bound[Start, -1, s] = Bound[Loop, 4, s] + Bound[Bubble, 3, s];
  Bound[Start, -2, s] = Bound[Double, 4, s] + Bound[Triangle, 4, s];

  Do[
   Bound[P1, a, s] = Sum[Bound[Start, c, s] × Bound[A, c, a, s], {c, {-2, -1, 0}}];
   , {a, {-2, -1, 0, 1, 2}}];
  , {s, {i, o}}];
```

For the weighted diagram we use the same cases. For $\underline{b} \neq 0$ we have to split the weight on $0 \longleftrightarrow x$ into $0 \longleftrightarrow \underline{b}$ and $\underline{b} \longleftrightarrow x$, which produces an extra factor 2.

```
In[3143]:= Do[Do[
    aprime = -Abs[a];
    Bound[DeltaStart, a, s] =
     Bound[C, 0, a, s] +
      If[aprime == 0, Bound[WeightedDouble, 1, s], 2 Bound[WeightedBubble, 1, s]] ×
       Bound[A, aprime, 0, s] +
      If[aprime == 0, 1, 2] × (Bound[Loop, 4, s] + Bound[Bubble, 4, s]/2) ×
       Bound[C, aprime, 0, s] (*b=v≠0*) +
      2 Bound[Loop, 4, s] × Bound[A, aprime, -1, s] +
      2 Bound[Loop, 4, s] × Bound[C, aprime, -1, s] (*dA(v,b)=1,v=0 *) +
      2 Bound[WeightedBubble, 2, s]/(2 d gjz[s]) Bound[A, aprime, -1, s] +
      2 Bound[Bubble, 3, s] × Bound[C, aprime, -1, s] (*dA(v,b)=1,v≠0 *) +
      2 Bound[WeightedBubble, 2, s] × Bound[A, aprime, -2, s] +
      2 Bound[Bubble, 4, s]/2 Bound[C, aprime, -2, s] (*dA(v,b)≥ 2,v=0*) +
      2 Bound[WeightedOpenBubble, 1, s] × Bound[A, aprime, -2, s] +
      2 Bound[Triangle, 4, s] × Bound[C, aprime, -2, s] (*dA(v,b)≥ 2,v≠0*);
   , {a, {-2, -1, 0, 1, 2}}];
  , {s, {i, o}}];
```

Initial Iota Block without weight

Here we define the bound of the initial part of the coefficients $\Xi^{(N),\iota}$ and $\Pi^{(N),\iota,\kappa}$, as given in Appendix C.4.1 of (II).

```
In[3144]:= Do[
  Do[
   Bound[B, 1, a, s] = Bound[tG, {1}, 1, s] × Bound[A, 0, a, s];
   , {a, {-2, -1, 0, 1, 2}}];
  Bound[B, 2, 0, s] = 1/(2 d) Bound[tG, {1}, 1, s]/gjz[s]
    (Bound[Bubble, 3, s] + Bound[Triangle, 3, s]);
  Bound[B, 2, -1, s] = Bound[tG, {1}, 1, s]/gjz[s] Bound[Triangle, 3, s]/(2 d);
  Bound[B, 2, -2, s] = Bound[tG, {1}, 1, s]/gjz[s] Bound[Square, 4, s]/(2 d);
  Bound[B, 2, 1, s] = Bound[B, 2, -1, s];
  Bound[B, 2, 2, s] = Bound[B, 2, -2, s];
  Bound[B, 3, 0, s] =
   Bound[Bubble, 3, s]/(2 d gjz[s]) (Bound[OpenBubble, 2, s] + Bound[OpenTriangle, 2, s]);
  Bound[B, 3, -1, s] = Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenTriangle, 2, s];
  Bound[B, 3, -2, s] = Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenSquare, 3, s];
  Bound[B, 3, 1, s] = Bound[B, 3, -1, s];
```

```
Bound[B, 3, 2, s] = Bound[B, 3, -2, s];
Bound[B, 4, 0, s] =
  1/(2 d) (Bound[Loop, 4, s] + 2 Bound[Bubble, 3, s] + Bound[Triangle, 3, s]);
Bound[B, 4, 1, s] =
  1/(2 d) (Bound[Loop, 4, s] + 2 Bound[Bubble, 3, s] + Bound[Triangle, 4, s]);
Bound[B, 4, 2, s] =
  1/(2 d) (Bound[Bubble, 3, s] + 2 Bound[Triangle, 4, s] + Bound[Square, 5, s]);
Bound[B, 4, -1, s] =
  1/(2 d) (Bound[Loop, 4, s] + Bound[Bubble, 4, s] + Bound[Triangle, 3, s]);
Bound[B, 4, -2, s] = 1/(2 d) (2 Bound[Triangle, 4, s] + Bound[Square, 5, s]);

SausageTobb = Bound[Double, 4, s];
(* a very crude bound on Σιϰ B2,2(eι+eϰ)*)
SausageWithPointTobb = Bound[Bubble, 4, s]/(2 d) Bound[tG, max, 1, s];
(* a double connection, with a line to eι*)

Bound[B, 5, 0, s] =
  gjz[s] (SausageWithPointTobb Bound[OpenTriangle, 0, s] +
     SausageTobb (Bound[Bubble, 3, s]/(2 d gjz[s]) + Bound[Triangle, 2, s]/(2 d gjz[s]) +
        Bound[Triangle, 1, s]/(2 d gjz[s])));
Bound[B, 5, 1, s] =
  gjz[s] (SausageWithPointTobb Bound[OpenTriangle, 1, s] +
     2 SausageTobb Bound[Triangle, 3, s]/(2 d gjz[s]));
Bound[B, 5, -1, s] = Bound[B, 5, 1, s];
Bound[B, 5, 2, s] =
  gjz[s] (SausageWithPointTobb Bound[OpenSquare, 2, s] +
     2 SausageTobb Bound[Square, 3, s]/(2 d gjz[s])) ;
Bound[B, 5, -2, s] = Bound[B, 5, 2, s];

Clear[SausageTobb, SausageWithPointTobb];

Do[
 Bound[B, 6, a, s] = Bound[tG, {1}, 1, s] × (Bound[P1, a, s] - Bound[A, 0, a, s])
  (*u=0≠b*);
```

```
Bound[B, 7, a, s] =
  (Bound[Loop, 4, s]/(2 d) (*v=e_1*) + (2 d - 1)/(2 d) Bound[Loop, 4, s] (*one step to bb *)
     Bound[tG, max, 2, s] + Bound[Double, 4, s] × Bound[tG, max, 1, s]
     (*more steps to bb*)) Bound[A, 0, a, s];
Bound[B, 8, a, s] =
 Bound[tG, {1}, 1, s] × Bound[tG, {1}, 3, s] (*bb=e1, v=0 *) Bound[A, -1, a, s] +
   Bound[Loop, 4, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) (*bb=e1, dA(bb,v)=1, v≠0*)
    Bound[A, -1, a, s] + Bound[Bubble, 4, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) (*bb=e1,
    dA(bb,v)>1, v≠0*) Bound[A, -2, a, s]
   + Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenBubble, 1, s] (*bb≠e1*) Bound[A, -2, a, s];
Bound[B, 9, a, s] =
  (Bound[Loop, 4, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[A, -1, a, s] (*v=e1,
     d(e1,bb)=1*) + Bound[Bubble, 4, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[A, -2, a, s]
     (*v=e1, d(e1,bb)>1*)) +
   (*v=u≠ei*) Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenBubble, 2, s] × Bound[A, -2, a, s];
Bound[B, 10, a, s] = Bound[Bubble, 3, s]/(2 d gjz[s])
   (Bound[tG, max, 2, s] × Bound[A, -1, a, s] +
     Bound[OpenBubble, 3, s] × Bound[A, -2, a, s]);
Bound[B, 11, a, s] = Bound[Bubble, 4, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[A, -1, a, s]
   (*u=e1,d(v,bb)=1*) + Bound[Triangle, 5, s] Bound[tG, {1}, 1, s]/(2 d gjz[s])
    Bound[A, -2, a, s] (*u=e1,d(v,bb)>1*) +
   Bound[Bubble, 3, s]/(2 d gjz[s])
    ( Bound[OpenBubble, 3, s] × Bound[A, -1, a, s] (*u≠e1,d(v,bb)=1*) +
      Bound[OpenTriangle, 4, s] × Bound[A, -2, a, s] (*u≠e1,d(v,bb)>1*));
Bound[B, 12, a, s] = Bound[B, 11, 0, s];
Bound[B, 13, a, s] =
  ((Bound[Bubble, 3, s]/(2 d gjz[s]))^2 (*u=e1*) + Bound[OpenBubble, 2, s] Bound[Triangle, 4, s]/(2 d gjz[s])
     (*u≠e1*)) Bound[A, -2, a, s];
 , {a, {-2, -1, 0, 1, 2}}];

Bound[B, 14, 0, s] =
```

```
(Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[tG, {1}, 3, s] +
    Bound[Triangle, 4, s]/(2 d gjz[s]) Bound[tG, max, 2, s]) Bound[OpenTriangle, 1, s];
Bound[B, 14, -1, s] =
  (Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[tG, {1}, 3, s] +
    Bound[Triangle, 4, s]/(2 d gjz[s]) Bound[tG, max, 2, s]) Bound[OpenTriangle, 2, s];
Bound[B, 14, -2, s] =
  (Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[tG, {1}, 3, s] +
    Bound[Triangle, 4, s]/(2 d gjz[s]) Bound[tG, max, 2, s]) Bound[OpenSquare, 3, s];
Bound[B, 14, 1, s] = Bound[B, 14, -1, s];
Bound[B, 14, 2, s] = Bound[B, 14, -2, s];

Bound[B, 15, 0, s] = Bound[Triangle, 3, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[tG, {1}, 3, s]
   (*u=e1, d(0,bb)=1*) + Bound[Square, 4, s] Bound[tG, {1}, 1, s]/(2 d gjz[s])
    Bound[tG, max, 2, s] (*u=e1, d(0,bb)>1*) +
   Bound[Double, 2, s] Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenTriangle, 1, s] (*u≠e1*);
Bound[B, 15, -1, s] =
  Bound[Triangle, 4, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[tG, {1}, 3, s] +
   Bound[Square, 5, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[tG, max, 2, s] +
   Bound[Double, 2, s] Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenTriangle, 2, s];
Bound[B, 15, -2, s] = Bound[Square, 5, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[tG, {1}, 3, s]
   (*u=e1, d(0,bb)=1*) + Bound[OpenSquare, 4, s] Bound[tG, {1}, 1, s]/(2 d gjz[s])
    Bound[Double, 4, s] (*u=e1, d(0,bb)>1*) +
   Bound[Double, 2, s] Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenSquare, 3, s];
Bound[B, 15, 1, s] = Bound[B, 15, -1, s];
Bound[B, 15, 2, s] = Bound[B, 15, -2, s];

Bound[B, 16, 0, s] =
  Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenBubble, 2, s] × Bound[OpenTriangle, 1, s] +
   Bound[Triangle, 3, s] × Bound[OpenBubble, 2, s] × Bound[OpenTriangle, 1, s];
```

```
Bound[B, 16, -1, s] =
  Bound[Bubble, 3, s]
  ------------------- Bound[OpenBubble, 2, s] × Bound[OpenTriangle, 2, s] +
     2 d gjz[s]
   Bound[Triangle, 3, s] × Bound[OpenBubble, 2, s] × Bound[OpenTriangle, 2, s];
Bound[B, 16, -2, s] =
  Bound[Bubble, 3, s]
  ------------------- Bound[OpenBubble, 2, s] × Bound[OpenSquare, 3, s] +
     2 d gjz[s]
   Bound[Triangle, 3, s] × Bound[OpenBubble, 2, s] × Bound[OpenSquare, 3, s];
Bound[B, 16, 1, s] = Bound[B, 16, -1, s];
Bound[B, 16, 2, s] = Bound[B, 16, -2, s];

Do[
 Bound[P1Iota, a, s] = Sum[Bound[B, c, a, s], {c, 1, 16}];
 , {a, {-2, -1, 0, 1, 2}}];
, {s, {i, o}}];
```

In[3145]:=
```
Labeled[
  Grid[{Join[{"Part"}, Table[t, {t, 1, 16}]],
    Join[{"Abs."}, Table[NumberForm[Bound[B, t, 1, o], 3], {t, 1, 16}]],
    Join[{"% of Total"}, Table[NumberForm[100 * Bound[B, t, 2, o] / Bound[P1Iota, 2, o], 3],
      {t, 1, 16}]]}, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Contribution to P1 ", Bold], Top] // Text
```

Out[3145]=

Contribution to P1

Part	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Abs.	0.00-04-64	0.00-02-67	0.00-00-38-1	0.00-04-1	6.12×10^{-7}	4.77×10^{-6}	1.15×10^{-6}	2.9×10^{-6}	1.3×10^{-6}	1.54×10^{-7}	1.65×10^{-6}	2.36×10^{-6}	3.62×10^{-7}	1.61 10
% of Total	38.8	22.2	3.4	34.2	0.08-32	0.411	0.09-59	0.255	0.114	0.01-34	0.143	0.121	0.03-23	0.01 44

Initial weighted Iota Block

Here we implement the bounds given in Appendix C.4.2 of (II). We begin the initial pieces in which $b = 0$.

In[3146]:=
```
Do[Do[
  Bound[D, 1, 0, a, s] = Bound[tG, {1}, 1, s] × Bound[C, 0, a, s];
  Bound[D, 1, ei, a, s] =
   2 Bound[tG, {1}, 1, s] (Bound[C, 0, a, s] + gj[s] × Bound[A, a, 0, s]);
                          1
  Bound[D, 2, 0, a, s] = --- Bound[C, 0, a, s];
                         2 d
  Bound[D, 3, 0, a, s] = Bound[tG, {1}, 3, s] × Bound[C, -1, a, s] +
    Bound[Bubble, 4, s]
    ------------------- Bound[C, -2, a, s];
        2 d gjz[s]
```

```
  Bound[D, 3, ei, a, s] =
   2 Bound[D, 3, 0, a, s] +
    2 gj[s] × (Bound[tG, max, 2, s] × Bound[A, a, -1, s] +
       2 Bound[tG, max, 1, s] × Bound[A, a, -2, s]);
  Bound[D, 4, 0, a, s] = 1/(2 d) Bound[C, 0, a, s];
  , {a, {-2, -1, 0, 1, 2}}];
 Bound[D, 1, ei, 0, s] = Bound[tG, {1}, 1, s] ( Bound[C, 0, a, s] + Bound[C, a, 0, s]);
 Bound[D, 2, 0, 0, s] = (gjz[s] + Bound[tG, {1}, 1, s]) × Bound[tG, {1}, 3, s]
   (*x=e1*) + 1/(2 d) Bound[tG, {1}, 1, s]/gjz[s]
    (2 Bound[WeightedBubble, 2, s] (*w=e1≠x or w=x≠e1*) +
      Bound[WeightedTriangle, 3, s] ) (*e1≠w≠x or w=x≠e1*);
 (*For M=1 && a=0 we can use symmetry to remove the factor 2,
 created when splitting the weight ||x-eι||2^2*)
 Bound[D, 1, ei, 0, s] = Bound[tG, {1}, 1, s] × Bound[C, 0, 0, s] +
   Bound[tG, {1}, 1, s] × Bound[A, 0, 0, s];
 Bound[D, 1, ei, -1, s] = 2 Bound[tG, {1}, 1, s] × Bound[C, 0, -1, s] +
   2 Bound[tG, {1}, 1, s]
    (Bound[Loop, 4, s]/(2 d gjz[s]) + Bound[Bubble, 4, s]/(2 d gjz[s]) + Bound[Triangle, 3, s]/(2 d gjz[s]));
 Bound[D, 1, ei, -2, s] = 2 Bound[tG, {1}, 1, s] × Bound[C, 0, -2, s] +
   2 Bound[tG, {1}, 1, s] × (Bound[OpenBubble, 2, s] + Bound[OpenTriangle, 2, s]);
 Bound[D, 1, ei, 1, s] = Bound[D, 1, ei, -1, s];
 Bound[D, 1, ei, 2, s] = Bound[D, 1, ei, -2, s];
 Bound[D, 2, ei, 0, s] = 1/(2 d) Bound[WeightedTriangle, 0, s];
 Bound[D, 2, ei, -1, s] = 1/(2 d) Bound[WeightedTriangle, 1, s]/(2 d gjz[s]);
 Bound[D, 2, ei, -2, s] = 1/(2 d) Bound[WeightedOpenTriangle, 0, s];
 Bound[D, 2, ei, 1, s] = Bound[D, 2, ei, -1, s];
 Bound[D, 2, ei, 2, s] = Bound[D, 2, ei, -2, s];

 Bound[D, 4, ei, 0, s] = 1/(2 d) Bound[WeightedTriangle, 1, s];
 Bound[D, 4, ei, -1, s] = 1/(2 d) Bound[WeightedTriangle, 2, s]/(2 d gjz[s]);
 Bound[D, 4, ei, -2, s] = 1/(2 d) Bound[WeightedOpenTriangle, 1, s];
 Bound[D, 4, ei, 1, s] = Bound[D, 4, ei, -1, s];
 Bound[D, 4, ei, 2, s] = Bound[D, 4, ei, -2, s];
 , {s, {i, o}}];
```

Then, we define the diagram for b≠0. These are trivial for lattice trees, but not for lattice animals.

```
In[3147]:= Do[
 SausageTobb = Bound[Double, 4, s];
```

```
(* a very crude bound on Σ_ικ B_{2,2}(e_ι+e_κ)*)
SausageWithPointTobb = Bound[Bubble, 4, s]/(2 d) Bound[tG, max, 1, s];
(* a double connection, with a line to e_ι*)


Bound[D, 5, ei, 0, s] =
  (2 SausageTobb Bound[WeightedTriangle, 1, s]/(2 d) (*v=b and v=0*) +
    Bound[WeightedOpenTriangle, 0, s] × gjz[s] SausageWithPointTobb);
Bound[D, 5, 0, 0, s] =
  (SausageTobb Bound[WeightedTriangle, 0, s]/(2 d) (*v=0*) +
    2 SausageTobb Bound[WeightedTriangle, 1, s]/(2 d) (*v=b, weight one *) +
    2 SausageTobb (Bound[Triangle, 2, s] + Bound[Bubble, 3, s])/(2 d) +
    Bound[WeightedOpenTriangle, 0, s]/(2 d) SausageWithPointTobb);
Bound[D, 5, ei, -1, s] =
  (2 SausageTobb Bound[WeightedTriangle, 2, s]/(2 d (2 d gjz[s])) (*v=b and v=0*) +
    Bound[WeightedOpenTriangle, 1, s]/(2 d) SausageWithPointTobb);
Bound[D, 5, 0, -1, s] =
  (SausageTobb Bound[WeightedTriangle, 1, s]/(2 d (2 d gjz[s])) (*v=0*) +
    2 SausageTobb Bound[WeightedTriangle, 2, s]/(2 d (2 d gjz[s])) (*v=b, weight one *) +
    2 SausageTobb (Bound[Triangle, 3, s] + Bound[Bubble, 4, s])/(2 d (2 d gjz[s])) +
    Bound[WeightedOpenTriangle, 1, s]/(2 d (2 d gjz[s])) SausageWithPointTobb);

Bound[D, 5, ei, -2, s] =
  (gjz[s] SausageTobb Bound[WeightedOpenTriangle, 0, s] +
    SausageTobb Bound[WeightedOpenTriangle, 1, s]/(2 d) (*v=0*) +
    gjz[s] SausageWithPointTobb Bound[WeightedOpenTriangle, 0, s]);
```

```
Bound[D, 5, 0, -2, s] =
  (2 SausageTobb Bound[WeightedOpenTriangle, 0, s] / (2 d) (*v=0 and v=b*) +
    SausageWithPointTobb Bound[WeightedOpenTriangle, 0, s] / (2 d)
     (1 + Bound[tG, max, 1, s]));

Bound[D, 5, ei, 1, s] = Bound[D, 5, ei, -1, s];
Bound[D, 5, ei, 2, s] = Bound[D, 5, ei, -2, s];
Bound[D, 5, 0, 1, s] = Bound[D, 5, 0, -1, s];
Bound[D, 5, 0, 2, s] = Bound[D, 5, 0, -2, s];
Clear[SausageTobb, SausageWithPointTobb];

Do[
 aprime = -Abs[a];
 Bound[D, 6, 0, a, s] =
  Bound[tG, {1}, 1, s] × (Bound[DeltaStart, a, s] - Bound[C, 0, a, s]);
 Bound[D, 6, ei, a, s] = 3/2 Bound[D, 6, 0, a, s] +
   3 Bound[tG, {1}, 1, s] ×
    (Bound[Double, 2, s] × Bound[A, a, 0, s] +
      (Bound[tG, {1}, 3, s] + Bound[Bubble, 3, s] / (2 d gjz[s])) × Bound[A, a, -1, s] +
      (Bound[tG, max, 2, s] + Bound[OpenBubble, 2, s]) × Bound[A, a, -2, s]) ;
 Bound[D, 7, 0, a, s] =
  2 (Bound[Loop, 4, s] / (2 d) (*v=e_1*) + Bound[Loop, 4, s] (*one step*)
       Bound[tG, max, 2, s] + Bound[Double, 4, s] × Bound[tG, max, 1, s]
      (*more steps*)) Bound[C, 0, a, s] +
   2 (Bound[Loop, 4, s] / (2 d) (*v=e_1*) + Bound[WeightedBubble, 2, s] / (2 d gjz[s])
       Bound[tG, max, 2, s] (*more steps*)) Bound[A, a, 0, s];

 Bound[D, 7, ei, a, s] =
  (Bound[Loop, 4, s] / (2 d) (*v=e_1*) + 2 Bound[Loop, 4, s] (*one step*)
       Bound[tG, max, 2, s] + 2 Bound[Double, 4, s] × Bound[tG, max, 1, s]
      (*more steps*)) Bound[C, 0, a, s] +
   2 ((2 d - 1) / (2 d) Bound[Loop, 4, s] (*one step*) Bound[tG, max, 2, s] +
```

```
      Bound[WeightedBubble, 3, s]
      ─────────────────────────── Bound[tG, max, 2, s] (*more steps*))
             2 d gjz[s]
    Bound[A, a, 0, s];
Bound[D, 8, ei, a, s] = Bound[Loop, 4, s] Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[C, -1, a, s]
   (*bb=e1, d(u,v)=1*) + Bound[Bubble, 3, s] Bound[tG, {1}, 1, s] / (2 d gjz[s]) (*bb=e1,
    d(u,v)>1 *) Bound[C, -2, a, s] +
   2
    (Bound[Bubble, 3, s] / (2 d gjz[s]) (Bound[OpenBubble, 1, s] × Bound[C, -1, a, s] +
        Bound[OpenBubble, 2, s] × Bound[C, -2, a, s]) (* weight right *) +
     (Bound[Loop, 4, s] / (2 d gjz[s]) Bound[A, a, -1, s] +
       Bound[WeightedBubble, 3, s] / (2 d gjz[s]) Bound[A, a, -2, s]) (* weigth left v==0*) +
      Bound[WeightedBubble, 3, s] / (2 d gjz[s]) Bound[tG, max, 1, s] ×
         (Bound[A, a, -1, s] + Bound[A, a, -2, s]) (*v≠0*)) (*bb≠e1*);

Bound[D, 8, 0, a, s] =
  2 Bound[tG, {1}, 1, s] × Bound[tG, {1}, 3, s] × Bound[C, -1, a, s] +
   2 Bound[tG, {1}, 3, s] Bound[tG, {1}, 1, s] / gjz[s] Bound[A, a, -1, s] (*bb=e1,
   v=0 *) +
   2 (Bound[Loop, 4, s] Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[C, -2, a, s] +
      Bound[Bubble, 4, s] Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[C, -2, a, s] +
      Bound[Loop, 4, s] Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[A, -1, a, s] +
      Bound[tG, max, 2, s] Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[A, -2, a, s])
     (*bb=e1, v≠0*) +
   2
    (Bound[Bubble, 3, s] / (2 d gjz[s]) (Bound[tG, max, 1, s] × Bound[C, -1, a, s] +
        Bound[OpenBubble, 2, s] × Bound[C, -2, a, s]) (* weight right *) +
     (Bound[Loop, 4, s] / (2 d gjz[s]) Bound[A, a, -1, s] +
       Bound[WeightedBubble, 3, s] / (2 d gjz[s]) Bound[A, a, -2, s]) (* weigth left v==0*) +
      Bound[WeightedBubble, 3, s] / (2 d gjz[s]) Bound[tG, max, 1, s] ×
```

```
        (Bound[A, a, -1, s] + Bound[A, a, -2, s]) (*v≠0*)) (*bb≠e1*);
Bound[D, 9, 0, a, s] =
 2 (Bound[Loop, 4, s] Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[C, -1, a, s] +
     Bound[tG, {1}, 1, s]
      (2 × (2 d - 2) Bound[tG, ikNotUsingi, 2, s] + 4 Bound[tG, twoiNotusingi, 4, s])
      Bound[A, a, -1, s] / (2 d)) (*v=u=ei,d(bb,v)=1*) +
  2 Bound[Bubble, 4, s] Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[C, -2, a, s] +
  2 (Bound[WeightedBubble, 3, s] Bound[tG, {1}, 1, s] / (2 d gjz[s])) (*v=u=ei,
   d(bb,v)>1*) Bound[Abar, -2, a, s] +
  2 Bound[Bubble, 3, s] / (2 d gjz[s]) Bound[OpenBubble, 2, s] × Bound[C, -2, a, s] +
  2 Bound[WeightedBubble, 2, s] / (2 d gjz[s]) Bound[tG, max, 1, s] × Bound[A, a, -1, s] +
  2 Bound[WeightedOpenBubble, 1, s] × Bound[tG, max, 1, s] × Bound[A, a, -2, s];
Bound[D, 9, ei, a, s] =
 2 (Bound[Loop, 4, s] Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[C, -1, a, s] +
     Bound[Loop, 4, s] Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[A, a, -1, s] / (2 d gjz[s])) (*v=u=ei,
  d(bb,v)=1*) +
  2 Bound[Bubble, 4, s] Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[C, -2, a, s] +
  2 (Bound[WeightedBubble, 2, s] Bound[tG, {1}, 1, s] / (2 d gjz[s])) (*v=u=ei,
   d(bb,v)>1*) Bound[Abar, -2, a, s] +
  3 Bound[Bubble, 3, s] / (2 d gjz[s]) Bound[OpenBubble, 2, s] × Bound[C, -2, a, s] +
  3 (Bound[WeightedBubble, 2, s] / (2 d gjz[s]) Bound[tG, max, 1, s] × Bound[A, a, -1, s] +
     Bound[WeightedOpenBubble, 1, s] × Bound[tG, max, 1, s] × Bound[A, a, -2, s]) +
  3 (*v=u≠ei*) Bound[Bubble, 3, s] / (2 d gjz[s]) Bound[tG, max, 1, s] ×
   (Bound[A, a, -1, s] + Bound[A, a, -2, s]);
Bound[D, 10, 0, a, s] =
 2 Bound[Bubble, 3, s] / (2 d gjz[s]) Bound[tG, max, 2, s] ×
   (Bound[C, -1, a, s] + Bound[A, a, -1, s] / (2 d gjz[s])) (*d(bb,0)=1*) +
```

```
  2 (Bound[Bubble, 3, s])/(2 d gjz[s])
   (Bound[OpenBubble, 3, s] × Bound[C, -2, a, s] +
     Bound[WeightedOpenBubble, 1, s] × Bound[Abar, -2, a, s]);
Bound[D, 10, ei, a, s] =
  2 (Bound[Bubble, 3, s])/(2 d gjz[s]) Bound[tG, max, 2, s] ×
   (Bound[C, -1, a, s] + ((2 * (2 d - 2) + 4))/(2 d) (Bound[A, a, -1, s])/(2 d gjz[s])) +
  3 (Bound[Bubble, 3, s])/(2 d gjz[s]) Bound[OpenBubble, 3, s] × Bound[C, -2, a, s] +
  3 (Bound[Bubble, 3, s])/(2 d gjz[s]) Bound[WeightedOpenBubble, 2, s] × Bound[Abar, -2, a, s] +
  3 (Bound[WeightedBubble, 3, s])/(2 d gjz[s]) Bound[OpenBubble, 3, s] × Bound[Abar, -2, a, s];
Bound[D, 11, 0, a, s] =
  2 Bound[Bubble, 4, s] (Bound[tG, {1}, 1, s])/(2 d gjz[s]) Bound[C, -1, a, s] +
   2 Bound[WeightedBubble, 3, s] (Bound[tG, {1}, 1, s])/(2 d gjz[s]) (Bound[A, a, -1, s])/(2 d gjz[s])
   (*u=e1,d(v,bb)=1*) + 2 Bound[Triangle, 5, s] (Bound[tG, {1}, 1, s])/(2 d gjz[s])
    Bound[C, -2, a, s] + 2 Bound[WeightedOpenBubble, 2, s] (Bound[tG, {1}, 1, s])/(2 d gjz[s])
    Bound[A, a, -2, s] (*u=e1,d(v,bb)>1*) +
   2 ((Bound[Bubble, 3, s])/(2 d gjz[s]) Bound[OpenBubble, 3, s] × Bound[C, -1, a, s] +
     (Bound[Bubble, 3, s])/(2 d gjz[s]) Bound[WeightedOpenBubble, 2, s] (Bound[A, a, -1, s])/(2 d gjz[s])
     (*u≠e1,d(v,bb)=1*) + (Bound[Bubble, 3, s])/(2 d gjz[s]) Bound[OpenTriangle, 4, s] ×
      Bound[C, -2, a, s] + (Bound[Bubble, 3, s])/(2 d gjz[s]) Bound[WeightedOpenBubble, 1, s] ×
      Bound[A, a, -2, s] (*u≠e1,d(v,bb)>1*));
Bound[D, 11, ei, a, s] =
  2 Bound[Bubble, 4, s] (Bound[tG, {1}, 1, s])/(2 d gjz[s]) Bound[C, -1, a, s] +
   2 Bound[WeightedBubble, 2, s] (Bound[tG, {1}, 1, s])/(2 d gjz[s]) (Bound[A, a, -1, s])/(2 d gjz[s]) +
   2 Bound[Triangle, 5, s] (Bound[tG, {1}, 1, s])/(2 d gjz[s]) Bound[C, -2, a, s] +
   2 Bound[WeightedOpenBubble, 1, s] (Bound[tG, {1}, 1, s])/(2 d gjz[s]) Bound[A, a, -2, s] +
```

```
 3 (Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenBubble, 3, s] × Bound[C, -1, a, s] +
    Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[WeightedOpenBubble, 2, s] Bound[A, a, -1, s]/(2 d gjz[s])
    (*u≠e1,d(v,bb)=1*) + Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenTriangle, 4, s] ×
     Bound[C, -2, a, s] + Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[WeightedOpenBubble, 1, s] ×
     Bound[A, a, -2, s] (*u≠e1,d(v,bb)>1*)) +
 3 Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenTriangle, 3, s] × Bound[Abar, a, -2, s];

Bound[D, 12, 0, a, s] =
 2 Bound[Bubble, 4, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[C, -1, a, s] +
  2 Bound[WeightedBubble, 2, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[A, a, -1, s]
  (*u=e1,d(v,bb)=1*) +
  2 (Bound[Triangle, 5, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[C, -2, a, s] +
     Bound[WeightedOpenBubble, 1, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[A, a, -2, s])
    (*u=e1,d(v,bb)>1*) +
  3 (Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenBubble, 3, s] × Bound[C, -1, a, s] +
     Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[WeightedOpenBubble, 2, s] Bound[A, a, -1, s]/(2 d gjz[s]) +
     Bound[WeightedBubble, 2, s]/(2 d gjz[s]) Bound[OpenBubble, 3, s] Bound[A, a, -1, s]/(2 d gjz[s]))
    (*u≠e1,d(v,bb)=1*) +
  3 (Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenTriangle, 4, s] × Bound[C, -2, a, s] +
     Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[WeightedOpenBubble, 1, s] × Bound[A, a, -2, s] +
     Bound[WeightedBubble, 2, s]/(2 d gjz[s]) Bound[OpenTriangle, 4, s] ×
      Bound[Abar, a, -2, s]);
Bound[D, 12, ei, a, s] =
 2 Bound[Bubble, 4, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[C, -1, a, s] +
  2 Bound[WeightedBubble, 3, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[A, a, -1, s]
  (*u=e1,d(v,bb)=1*) +
```

$$2\left(\text{Bound}\left[\text{Triangle, 5, s}\right]\frac{\text{Bound[tG, \{1\}, 1, s]}}{\text{2 d gjz[s]}}\text{Bound[C, -2, a, s]}+\right.$$

$$\left.\text{Bound}\left[\text{WeightedOpenBubble, 2, s}\right]\frac{\text{Bound[tG, \{1\}, 1, s]}}{\text{2 d gjz[s]}}\text{Bound[A, a, -2, s]}\right)$$

$$(*\text{u}=\text{e}_1,\text{d(v,bb)}>1*)+$$

$$3\left(\frac{\text{Bound[Bubble, 3, s]}}{\text{2 d gjz[s]}}\text{Bound[OpenBubble, 3, s]}\times\text{Bound[C, -1, a, s]}+\right.$$

$$\frac{\text{Bound[Bubble, 3, s]}}{\text{2 d gjz[s]}}\text{Bound}\left[\text{WeightedOpenBubble, 2, s}\right]\frac{\text{Bound[A, a, -1, s]}}{\text{2 d gjz[s]}}+$$

$$\left.\frac{\text{Bound}\left[\text{WeightedBubble, 2, s}\right]}{\text{2 d gjz[s]}}\text{Bound[OpenBubble, 3, s]}\frac{\text{Bound[A, a, -1, s]}}{\text{2 d gjz[s]}}\right)$$

$$(*\text{u}\neq\text{e}_1,\text{d(v,bb)}=1*)+$$

$$3\left(\frac{\text{Bound[Bubble, 3, s]}}{\text{2 d gjz[s]}}\text{Bound}\left[\text{OpenTriangle, 4, s}\right]\times\text{Bound[C, -2, a, s]}+\right.$$

$$\frac{\text{Bound[Bubble, 3, s]}}{\text{2 d gjz[s]}}\text{Bound}\left[\text{WeightedOpenBubble, 1, s}\right]\times\text{Bound[A, a, -2, s]}+$$

$$\frac{\text{Bound}\left[\text{WeightedBubble, 2, s}\right]}{\text{2 d gjz[s]}}\text{Bound}\left[\text{OpenTriangle, 4, s}\right]\times$$

$$\left.\text{Bound[Abar, a, -2, s]}\right);$$

$$\text{Bound[D, 13, 0, a, s]}=2\left(\frac{\text{Bound[Bubble, 3, s]}}{\text{2 d gjz[s]}}\right)^2\text{Bound[C, -2, a, s]}+$$

$$2\,\frac{\text{Bound[Bubble, 3, s]}}{\text{2 d gjz[s]}}\,\frac{\text{Bound}\left[\text{WeightedBubble, 2, s}\right]}{\text{2 d gjz[s]}}\text{Bound[Abar, a, -2, s]}$$

$$(*\text{u}=\text{e}_1*)+$$

$$2\,\text{Bound[OpenBubble, 2, s]}\frac{\text{Bound}\left[\text{Triangle, 4, s}\right]}{\text{2 d gjz[s]}}\text{Bound[C, -2, a, s]}+$$

$$2\,\text{Bound}\left[\text{WeightedOpenBubble, 1, s}\right]\frac{\text{Bound}\left[\text{Triangle, 4, s}\right]}{\text{2 d gjz[s]}}\text{Bound[Abar, a, -2, s]}$$

$$(*\text{u}\neq\text{e}_1*);$$

$$\text{Bound}\left[\text{D, 13, ei, a, s}\right]=2\left(\frac{\text{Bound[Bubble, 3, s]}}{\text{2 d gjz[s]}}\right)^2\text{Bound[C, -2, a, s]}+$$

$$2\,\frac{\text{Bound[Bubble, 3, s]}}{\text{2 d gjz[s]}}\,\frac{\text{Bound}\left[\text{WeightedBubble, 2, s}\right]}{\text{2 d gjz[s]}}\text{Bound[Abar, a, -2, s]}$$

$$(*\text{u}=\text{e}_1*)+$$

$$3\,\text{Bound[OpenBubble, 2, s]}\frac{\text{Bound}\left[\text{Triangle, 4, s}\right]}{\text{2 d gjz[s]}}\text{Bound[C, -2, a, s]}+$$

$$3\,\text{Bound}\left[\text{WeightedOpenBubble, 1, s}\right]\frac{\text{Bound}\left[\text{Triangle, 4, s}\right]}{\text{2 d gjz[s]}}\text{Bound[Abar, a, -2, s]}$$

$$(*\text{u}\neq\text{e}_1*)+3\,\text{Bound[OpenBubble, 2, s]}\frac{\text{Bound}\left[\text{Triangle, 4, s}\right]}{\text{2 d gjz[s]}}\text{Bound[Abar, -2, a, s]}$$

$$(*\text{u}\neq\text{e}_1*);$$

$$\text{Bound[D, 14, 0, a, s]}=$$

```
2 (Bound[Bubble, 3, s] / (2 d gjz[s]) Bound[tG, {1}, 3, s] +
     Bound[Triangle, 4, s] / (2 d gjz[s]) Bound[tG, max, 2, s])
  Bound[WeightedOpenTriangle, 0, s] +
 2 (Bound[Bubble, 3, s] / (2 d gjz[s]) Bound[tG, {1}, 3, s] +
     Bound[OpenBubble, 2, s] × Bound[WeightedBubble, 2, s]) Bound[Abar, a, -2, s];
Bound[D, 14, ei, a, s] =
 (2 Bound[Bubble, 3, s] / (2 d gjz[s]) Bound[tG, {1}, 3, s] +
     3 Bound[Triangle, 4, s] / (2 d gjz[s]) Bound[tG, max, 2, s])
  Bound[WeightedOpenTriangle, 0, s] +
 2 Bound[OpenBubble, 2, s] × Bound[WeightedBubble, 2, s] × Bound[Abar, a, -2, s] +
 (2 Bound[Bubble, 3, s] / (2 d gjz[s]) Bound[tG, {1}, 3, s] (2 × (2 d - 2) + 4) / (2 d) +
     3 Bound[Triangle, 4, s] / (2 d gjz[s]) Bound[tG, max, 2, s]) Bound[Abar, a, -2, s];

, {a, {-2, -1, 0, 1, 2}}];

Bound[D, 15, 0, 0, s] =
 2 Bound[WeightedOpenTriangle, 1, s] Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[Double, 2, s] +
  2 Bound[OpenTriangle, 2, s] Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[WeightedDouble, 1, s]
  (*u=e1*) + 2 Bound[WeightedOpenTriangle, 0, s] × Bound[Double, 2, s]
  Bound[Bubble, 3, s] / (2 d gjz[s]) (*u≠e1*) +
  2 Bound[OpenTriangle, 1, s] Bound[Bubble, 3, s] / (2 d gjz[s]) Bound[WeightedDouble, 1, s]
  (*u≠e1*);
(*For a=1,-1*)
Bound[D, 15, 0, -1, s] =
 2 Bound[WeightedOpenTriangle, 2, s] / (2 d gjz[s]) Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[Double, 2, s] +
  2 Bound[OpenTriangle, 3, s] / (2 d gjz[s]) Bound[tG, {1}, 1, s] / (2 d gjz[s]) Bound[WeightedDouble, 1, s]
  (*u=e1*) + 2 Bound[WeightedOpenTriangle, 1, s] / (2 d gjz[s]) Bound[Double, 2, s]
  Bound[Bubble, 3, s] / (2 d gjz[s]) (*u≠e1*) +
```

```
  2 Bound[OpenTriangle, 2, s]/(2 d gjz[s]) Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[WeightedDouble, 1, s]
  (*u≠e1*);

(*For a=2,-2*)
Bound[D, 15, 0, -2, s] =
  2 Bound[WeightedOpenTriangle, 1, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[Double, 2, s] +
   2 (2 d gjz[s])^1 VarGamma2[s]^3 K[3, 2, {0}] × Bound[tG, {1}, 1, s] ×
    Bound[WeightedDouble, 1, s] (*u=e1*) +
   2 Bound[WeightedOpenTriangle, 0, s] × Bound[Double, 2, s] Bound[Bubble, 3, s]/(2 d gjz[s])
   (*u≠e1*) + 2 (2 d gjz[s])^1 VarGamma2[s]^3 K[3, 1, {0}] Bound[Bubble, 3, s]/(2 d gjz[s])
    Bound[WeightedDouble, 1, s] (*u≠e1*);
Bound[D, 15, ei, 0, s] = 3/2 Bound[D, 15, 0, 0, s] + 3 Bound[B, 15, 0, s];
Bound[D, 15, ei, -1, s] = 3/2 Bound[D, 15, 0, -1, s] + 3 Bound[B, 15, -1, s]/(2 d gjz[s]);

Bound[D, 15, ei, -2, s] =
  3/2 Bound[D, 15, 0, -2, s] +
   3 (Bound[OpenTriangle, 3, s] Bound[tG, {1}, 1, s]/(2 d gjz[s]) Bound[tG, {1}, 3, s] +
      (2 d gjz[s])^2 VarGamma2[s]^3 K[3, 2, {0}] Bound[tG, {1}, 1, s]/(2 d gjz[s])
       Bound[Double, 4, s] + Bound[Double, 2, s] Bound[Bubble, 3, s]/(2 d gjz[s])
       (2 d gjz[s])^1 VarGamma2[s]^3 K[3, 1, {0}]);
Bound[D, 15, 0, 1, s] = Bound[D, 15, 0, -1, s];
Bound[D, 15, 0, 2, s] = Bound[D, 15, 0, -2, s];
Bound[D, 15, ei, 1, s] = Bound[D, 15, ei, -1, s];
Bound[D, 15, ei, 2, s] = Bound[D, 15, ei, -2, s];

Bound[D, 16, 0, 0, s] =
  2 (Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenBubble, 2, s] × Bound[WeightedOpenTriangle, 0, s]
      (*d(0,v)=1*) + Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[WeightedOpenBubble, 1, s] ×
       Bound[OpenTriangle, 1, s] (*d(0,v)=1*) +
      Bound[Triangle, 4, s]/(2 d gjz[s]) Bound[OpenBubble, 2, s] ×
       Bound[WeightedOpenTriangle, 0, s] (*d(0,v)>1*) +
```

```
    Bound[Triangle, 4, s]/(2 d gjz[s]) Bound[WeightedOpenBubble, 1, s] ×
      Bound[OpenTriangle, 1, s] (*d(0,v)>1*));

Bound[D, 16, 0, -1, s] =
  2 (Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenBubble, 2, s] Bound[WeightedOpenTriangle, 1, s]/(2 d gjz[s]) +
    Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[WeightedOpenBubble, 1, s] Bound[OpenTriangle, 2, s]/(2 d gjz[s]) +
    Bound[WeightedTriangle, 3, s] × Bound[OpenBubble, 2, s]
      Bound[OpenTriangle, 2, s]/(2 d gjz[s]) +
    Bound[Triangle, 4, s]/(2 d gjz[s]) Bound[WeightedOpenBubble, 1, s]
      Bound[WeightedOpenTriangle, 2, s]/(2 d gjz[s]));

Bound[D, 16, 0, -2, s] =
  2 (Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenBubble, 2, s] × Bound[WeightedOpenTriangle, 0, s] +
    Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[WeightedOpenBubble, 1, s] (2 d gjz[s])^1
      VarGamma2[s]^3 K[3, 1, {0}] +
    Bound[Triangle, 3, s] × Bound[OpenBubble, 2, s] ×
      Bound[WeightedOpenTriangle, 0, s] +
    Bound[Triangle, 4, s]/(2 d gjz[s]) Bound[WeightedOpenBubble, 1, s] (2 d gjz[s])^1
      VarGamma2[s]^3 K[3, 1, {0}]);

Bound[D, 16, ei, 0, s] = 3/2 Bound[D, 16, 0, 0, s] + 3 Bound[B, 16, 0, s];
Bound[D, 16, ei, -1, s] = 3/2 Bound[D, 16, 0, -1, s] + 3 Bound[B, 16, -1, s]/(2 d gjz[s]);
Bound[D, 16, ei, -2, s] =
  3/2 Bound[D, 16, 0, -2, s] +
   3 (Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenBubble, 2, s] +
      Bound[Triangle, 3, s] × Bound[OpenBubble, 2, s]) (2 d gjz[s])^1
    VarGamma2[s]^3 K[3, 1, {0}];


Bound[D, 16, 0, 1, s] = Bound[D, 16, 0, -1, s];
Bound[D, 16, 0, 2, s] = Bound[D, 16, 0, -2, s];
```

```
Bound[D, 16, ei, 1, s] = Bound[D, 16, ei, -1, s];
Bound[D, 16, ei, 2, s] = Bound[D, 16, ei, -2, s];

Do[
 Bound[D, 0, a, s] = Sum[Bound[D, c, 0, a, s], {c, 1, 16}];
 Bound[D, ei, a, s] = Sum[Bound[D, c, ei, a, s], {c, 1, 16}];
 , {a, {-2, -1, 0, 1, 2}}];
, {s, {i, o}}];
```

```
In[3148]:= Labeled[
  Grid[{Join[{"Part"}, Table[t, {t, 1, 8}]],
    Join[{"Abs."}, Table[NumberForm[Bound[D, t, 0, 2, o], 3], {t, 1, 8}]],
    Join[{"% of Total"}, Table[NumberForm[100 * Bound[D, t, 0, 2, o] / Bound[D, 0, 2, o], 3],
      {t, 1, 8}]]}, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Contribution to D^(I,2) ", Bold], Top] // Text
Labeled[
  Grid[{Join[{"Part"}, Table[t, {t, 9, 16}]],
    Join[{"Abs."}, Table[NumberForm[Bound[D, t, 0, 2, o], 3], {t, 9, 16}]],
    Join[{"% of Total"}, Table[NumberForm[100 * Bound[D, t, 0, 2, o] / Bound[D, 0, 2, o], 3],
      {t, 9, 16}]]}, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Contribution to D^(I,2) ", Bold], Top] // Text

Labeled[
  Grid[{Join[{"Part"}, Table[t, {t, 1, 8}]],
    Join[{"Abs."}, Table[NumberForm[Bound[D, t, ei, 2, o], 3], {t, 1, 8}]],
    Join[{"% of Total"}, Table[NumberForm[100 * Bound[D, t, ei, 2, o] / Bound[D, ei, 2, o], 3],
      {t, 1, 8}]]}, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Contribution to D^(II,2) ", Bold], Top] // Text
Labeled[
  Grid[{Join[{"Part"}, Table[t, {t, 9, 16}]],
    Join[{"Abs."}, Table[NumberForm[Bound[D, t, ei, 2, o], 3], {t, 9, 16}]],
    Join[{"% of Total"}, Table[NumberForm[100 * Bound[D, t, ei, 2, o] / Bound[D, ei, 2, o], 3],
      {t, 9, 16}]]}, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Contribution to D^(II,2) ", Bold], Top] // Text
```

Contribution to $D^{I,2}$

Out[3148]=

Part	1	2	3	4	5	6	7	8
Abs.	0.00844	0.00745	0.000691	0.00745	0.0000177	0.000263	0.0000435	0.00027
% of Total	32.4	28.6	2.65	28.6	0.0678	1.01	0.167	1.04

Contribution to $D^{I,2}$

Out[3149]=

Part	9	10	11	12	13	14	15	16
Abs.	0.000215	0.000175	0.000101	0.000175	0.000257	0.0000447	0.0000699	0.000415
% of Total	0.825	0.67	0.387	0.67	0.984	0.171	0.268	1.59

Contribution to $D^{II,2}$

Out[3150]=

Part	1	2	3	4	5	6	7	8
Abs.	0.0179	0.00745	0.00886	0.00361	0.0000139	0.000416	0.000032	0.000209
% of Total	44.	18.3	21.8	8.87	0.0342	1.02	0.0787	0.514

Contribution to $D^{II,2}$

Out[3151]=

Part	9	10	11	12	13	14	15	16
Abs.	0.000375	0.000148	0.000172	0.000147	0.000404	0.0000688	0.000133	0.000713
% of Total	0.922	0.363	0.422	0.362	0.994	0.169	0.327	1.75

Definition of the vectors and matrices

We condition on the length of the backbone and identify whether the backbone is on the top or bottom of the diagram.

index	interpretation
-2	backbone is on bottom, d (u, v) ≥ 2.
-1	backbone is on bottom, d (u, v) = 1.
0	u = v
1	backbone is on top, d (u, v) = 1.
2	backbone in on the top, d (u, v) ≥ 2.

In[3152]:=

```
Do[
  Matrix[A, s] = Table[Bound[A, a, b, s], {a, -2, 2}, {b, -2, 2}];
  Matrix[Abar, s] = Table[Bound[Abar, a, b, s], {a, -2, 2}, {b, -2, 2}];
  Matrix[C, s] = Table[Bound[C, a, b, s], {a, -2, 2}, {b, -2, 2}];

  Vector[P1, s] = Table[Bound[P1, a, s], {a, -2, 2}];
  Vector[DeltaStart, s] = Table[Bound[DeltaStart, a, s], {a, -2, 2}];
  Vector[EndOpen, s] = Table[Bound[A, b, 0, s], {b, -2, 2}];
  Vector[EndClosed, s] = Table[Bound[A, 0, b, s], {b, -2, 2}];
  Vector[DeltaEnd, s] = Table[Bound[C, b, 0, s], {b, -2, 2}];

  Vector[P1Iota, s] = Table[Bound[P1Iota, a, s], {a, -2, 2}];
  Vector[D, 0, s] = Table[Bound[D, 0, a, s], {a, -2, 2}];
  Vector[D, ei, s] = Table[Bound[D, ei, a, s], {a, -2, 2}];
  , {s, {i, o}}]
```

To compute the sum over matrices we compute a representation of the opening and closing vectors using eigenvalues of the matrices A. If the matrix is not invertible, then such a representation (using real values only) does not need to exist. We bypass this problem by using a symmetric matrix that dominates A, see *ASym* below, and use this in our bound for $N \geq 4$.

In[3153]:=
```
Do[
  Matrix[ASym, s] = Table[Max[Bound[A, a, b, s], Bound[A, b, a, s]], {a, -2, 2},
    {b, -2, 2}];

  EigenA[s] = Eigensystem[Transpose[Matrix[ASym, s]]];
  InverseProduct[left, s] = Inverse[Transpose[EigenA[s][[2]]]].Vector[P1, s];
  InverseProduct[iota, s] = Inverse[Transpose[EigenA[s][[2]]]].Vector[P1Iota, s];
  InverseProduct[right, s] = Inverse[Transpose[EigenA[s][[2]]]].Vector[EndClosed, s];
  Do[
   EigenVector[left, j, s] = EigenA[s][[2, j]] * InverseProduct[left, s][[j]];
   EigenVector[iota, j, s] = EigenA[s][[2, j]] * InverseProduct[iota, s][[j]];
   EigenVector[right, j, s] = EigenA[s][[2, j]] * InverseProduct[right, s][[j]];
   EigenValue[j, s] = EigenA[s][[1, j]];
   , {j, 1, 5}]

  , {s, {i, o}}]
```

Bounds on the coefficients

Bounds for N=0

Here, we implement the bounds stated in Lemma 5.1 of (II).

In[3154]:=
```
Do[
  Bound[Xi, 0, s] = 1/gLower Bound[Double, 2, s];
  Bound[Xi, 0, Delta, s] = Bound[WeightedDouble, 1, s];
  Bound[Xi, R, 0, s] = 1/gLower Bound[Double, 4, s];
  Bound[Xi, R, 0, Delta, s] = Bound[WeightedDouble, 2, s];
  Bound[Psi, RII, 0, s] = Bound[Xi, R, 0, s];
  Bound[Psi, RII, 0, Delta, s] = Bound[Xi, R, 0, Delta, s];
  Bound[Psi, RI, 0, s] =
   muOverMub[s] ×
    (Bound[Loop, 4, s] (*x=e_1 does no contribute*) + Bound[Loop, 4, s]
      (*x=e_κ+e_1 one con. with two steps is excluded*) + Bound[Bubble, 6, s]/2);
  Bound[Psi, RI, 0, Delta, s] = Bound[Xi, 0, Delta, s] + Bound[Psi, RI, 0, s];
  , {s, {i, o}}]
```

Next, we implement the bounds on $\Xi^{(0),\iota}$ stated in Lemma 5.2 of (II):

In[3155]:=
```
Do[
  conAlpha[n_, m_] :=
   1/gLower ((*u=0*) Bound[tG, {1}, 1, s] × Bound[Double, 2 * n, s] +
      If[n == 0, Bound[Double, 2, s], 0] (*x=e_ι*) + (*u=e_ι≠x*)
```

```
      1/(2 d) Bound[tG, {1}, 1, s]/gjz[s] Bound[Bubble, 1 + n + Max[1, m], s] +
      1/gLower (If[n ≤ 1, Bound[tG, max, 2, s] (2 d - 1) gjz[s] × Bound[tG, {1}, 3, s], 0] +
         Bound[tG, max, 1, s] × Bound[Double, 4, s]) (*u=x≠{0,e1}*) +
      Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenBubble, n + 1, s] (*u≠0,e1,x and 0≠x*));

conBeta[n_] :=
  1/gLower (Bound[tG, {1}, 1, s] × Bound[WeightedDouble, n + 1, s] +
      If[n == 0, Bound[tG, {1}, 1, s] × Bound[tG, {1}, 3, s] (*u=e1=x*), 0] +
      Bound[tG, {1}, 1, s] Bound[WeightedDouble, 2, s]/(2 d gjz[s]) (*u=e1≠x*)) +
   1/gLower (If[n == 0, Bound[Loop, 4, s]/(2 d gjz[s]) Bound[tG, {1}, 3, s] (*u=x, d(0,x)=1*), 0] +
      Bound[WeightedBubble, 2, s]/(2 d gjz[s]) Bound[tG, max, 2, s] (*u=x, d(0,x)≥ 2*) +
      Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[WeightedOpenBubble, 1, s]);
conGamma[n_] :=
  (1/gLower Bound[tG, {1}, 1, s] × Bound[Double, 2, s] +
     Bound[tG, {1}, 1, s] × Bound[WeightedDouble, 1, s]) (*u=0*) +
   If[n == 1, Bound[tG, max, 1, s]
     (Bound[tG, {2}, 2, s]^2 (2 d - 2) + Bound[tG, {0, 1}, 4, s]^2) (*u=x*), 0] +
   Bound[tG, max, 2, s] Bound[WeightedBubble, 2, s]/(2 d gjz[s]) (*u=x*) +
   2 (Bound[WeightedOpenBubble, 1, s] Bound[Bubble, 3, s]/(2 d gjz[s]) +
      1/gLower Bound[Bubble, 3, s]/(2 d gjz[s]) Bound[OpenBubble, 2, s]) (*u≠ei*);
Bound[XiIota, 0, s] = muOverMub[s] × Bound[tG, {1}, 1, s] + conAlpha[1, 0];
Bound[XiIota, RI, 0, s] = conAlpha[1, 2];
Bound[XiIota, RII, 0, s] = conAlpha[2, 1];
Bound[XiIota, 0, Delta, 0, s] = conBeta[0];
Bound[XiIota, 0, Delta, ei, s] = conGamma[1];
Bound[XiIota, RI, 0, Delta, ei, s] = conGamma[2];
Bound[XiIota, RII, 0, Delta, 0, s] = conBeta[1];
Bound[XiIota, alphaI, 0, Atei, s] = 1/gLower Bound[Double, 2, s];
Bound[XiIota, alphaII, 0, AtZero, s] = muOverMub[s] × Bound[tG, {1}, 1, s];

Bound[XiIota, alphaI, 0, SumAroundei, s] =
```

```
muOverMub[s] × Bound[tG, {1}, 1, s] +
  1/gLower
   (Bound[Double, 4, s] × gjz[s] + 1/(2 d) Bound[tG, {1}, 1, s]/gjz[s] Bound[Loop, 4, s]);
Bound[XiIota, alphaII, 0, SumAroundZero, s] =
 1/gLower Bound[tG, {1}, 1, s] × Bound[tG, {1}, 3, s] +
  1/gLower 1/(2 d) Bound[tG, {1}, 1, s]/gjz[s] Bound[Loop, 4, s] +
  1/gLower Bound[tG, {1}, 1, s]/gjz[s] Bound[Bubble, 3, s] × Bound[tG, max, 1, s] +
  Bound[Loop, 4, s] × (Bound[tG, {1}, 1, s] + Bound[tG, ikNotUsingi, 2, s]);
Bound[Pi, alpha, 0, s] = (2 d - 1) Bound[Double, 2, s] (*x=e1*) z[s];
Bound[Pi, R, 0, s] = (2 d - 1) gjz[s] × Bound[tG, {1}, 1, s] +
  gjz[s] × conAlpha[1, 1];
Bound[Pi, R, 0, Delta, eiek, s] =
 8 d^2 gjz[o] × Bound[tG, {1}, 1, s] + (2 d)^2 gjz[o] × conGamma[1] +
  4 d (d + 1) gjz[o] × conAlpha[1, 1];
, {s, {i, o}}];
```

Bounds for *N=1*

Implementation of the bounds stated in Lemma 5.3 of (II).

```
In[3156]:= Do[
  Bound[Xi, 1, s] = muOverMub[s] × Bound[P1, 0, s];
  Bound[Xi, R, 1, s] =
   muOverMub[s] ×
    (Bound[P1, 0, s] - Bound[A, 0, 0, s] (*b≠0*) + 2 Bound[Bubble, 4, s]
      (*b=0,w in 0,x *) + Bound[Triangle, 4, s]);

  Bound[Psi, RI, 1, s] =
   muOverMub[s] ×
    (Bound[P1, 0, s] - Bound[A, 0, 0, s] (*b≠0*) + 4 Bound[Loop, 4, s] +
      2 Bound[Bubble, 5, s] + Bound[Bubble, 3, s] + Bound[Bubble, 4, s] +
      Bound[Triangle, 5, s]);
  Bound[Psi, RII, 1, s] =
   muOverMub[s] ×
    (Bound[P1, 0, s] - Bound[A, 0, 0, s] (*b≠0*) + 2 Bound[Bubble, 4, s]
      (*b=0,w in 0,x *) + Bound[Triangle, 4, s]);

  Bound[Xi, 1, Delta, s] = Bound[DeltaStart, 0, s];
  Bound[Xi, R, 1, Delta, s] = Bound[DeltaStart, 0, s];
  Bound[Psi, RI, 1, Delta, s] = Bound[Xi, 1, Delta, s] + Bound[Psi, RI, 1, s];
  Bound[Psi, RII, 1, Delta, s] = Bound[Xi, R, 1, Delta, s];

  Bound[XiIota, 1, s] = muOverMub[s] × Bound[P1Iota, 0, s];
  Bound[XiIota, 1, Delta, 0, s] = Bound[D, 0, 0, s];
  Bound[XiIota, 1, Delta, ei, s] = Bound[D, ei, 0, s];
  , {s, {i, o}}];
```

Bounds for $N = 2, 3$

We consider also $N=2,3$ as a special case, as these are large enough to be of numerical significance. We first implement the bounds on the absolute value of the coefficient as stated in Proposition 5.4 of (II).

```
In[3157]:= Do[
  Bound[Xi, 2, s] = muOverMub[s] × Vector[P1, s].Vector[EndOpen, s];
  Bound[Xi, 3, s] = muOverMub[s] × Vector[P1, s].Matrix[A, s].Vector[EndOpen, s];

  Bound[XiIota, 2, s] = muOverMub[s] × Vector[P1Iota, s].Vector[EndOpen, s];
  Bound[XiIota, 3, s] =
   muOverMub[s] × Vector[P1Iota, s].Matrix[A, s].Vector[EndOpen, s];
  , {s, {i, o}}];
```

We improve the bound on the weighted diagrams stated in Proposition 5.4 of (II) slightly. The bound stated in the proposition splits the weight $|x|_2^2$ using

$$|x|_2^2 \;\le\; 2\,|y|_2^2 + 2\,|x-y|_2^2$$

when splitting the weight along the different building blocks, see Figure 14 of (II). In the case that the shared lines collapse to a point (u=y, so that l=0) we can use

$$|x|_2^2 = |y|_2^2 + |x-y|_2^2 + 2\sum_i x_i\, y_i$$

By the spatial symmetry in all directions of the building blocks, that only exists in the full extent if the shared line is collapsed, the sum of $x_i\, y_i$ cancels out. Thus, the bounds stated in Proposition 5.4 of (II) are a factor 2 too big for this specific case. This improves the bound by around 10 percent. This is analogous to the improvement explained in (6.22) of (II).

```
In[3158]:= Do[
    Bound[Xi, 2, Delta, s] =
     2 (Vector[P1, s].Vector[DeltaEnd, s] +
         Vector[DeltaStart, s].Vector[EndClosed, s]) -
      (Vector[P1, s][[3]] × Vector[DeltaEnd, s][[3]] +
        Vector[DeltaStart, s][[3]] × Vector[EndClosed, s][[3]]);
    Bound[Xi, 3, Delta, s] =
     3 (Vector[P1, s].Matrix[A, s].Vector[DeltaEnd, s] +
         Vector[P1, s].Matrix[C, s].Vector[EndClosed, s] +
         Vector[DeltaStart, s].Matrix[A, s].Vector[EndClosed, s]) -
      Sum[Vector[P1, s][[3]] × Matrix[A, s][[3, b]] × Vector[DeltaEnd, s][[b]] +
        Vector[P1, s][[3]] × Matrix[C, s][[3, b]] × Vector[EndClosed, s][[b]] +
        Vector[DeltaStart, s][[3]] × Matrix[A, s][[3, b]] × Vector[EndClosed, s][[b]]
       , {b, 1, 5}] -
      Sum[Vector[P1, s][[a]] × Matrix[A, s][[a, 3]] × Vector[DeltaEnd, s][[3]] +
        Vector[P1, s][[a]] × Matrix[C, s][[a, 3]] × Vector[EndClosed, s][[3]] +
        Vector[DeltaStart, s][[a]] × Matrix[A, s][[a, 3]] × Vector[EndClosed, s][[3]]
       , {a, 1, 5}];

    Do[
     Bound[XiIota, 2, Delta, type, s] =
      2 (Vector[P1Iota, s].Vector[DeltaEnd, s] +
          Vector[D, type, s].Vector[EndClosed, s]) -
       (Vector[P1Iota, s][[3]] × Vector[DeltaEnd, s][[3]] +
         Vector[D, type, s][[3]] × Vector[EndClosed, s][[3]]);
     Bound[XiIota, 3, Delta, type, s] =
      Bound[XiIota, 3, Delta, type, s] =
       3 (Vector[P1Iota, s].Matrix[A, s].Vector[DeltaEnd, s] +
           Vector[P1Iota, s].Matrix[C, s].Vector[EndClosed, s] +
           Vector[D, type, s].Matrix[A, s].Vector[EndClosed, s]) -
        Sum[Vector[P1Iota, s][[a]] × Matrix[A, s][[a, 3]] × Vector[DeltaEnd, s][[3]] +
          Vector[P1Iota, s][[a]] × Matrix[C, s][[a, 3]] × Vector[EndClosed, s][[3]] +
          Vector[D, type, s][[a]] × Matrix[A, s][[a, 3]] × Vector[EndClosed, s][[3]],
         {a, 1, 5}] -
        Sum[Vector[P1Iota, s][[3]] × Matrix[A, s][[3, b]] × Vector[DeltaEnd, s][[b]] +
          Vector[P1Iota, s][[3]] × Matrix[C, s][[3, b]] × Vector[EndClosed, s][[b]] +
          Vector[D, type, s][[3]] × Matrix[A, s][[3, b]] × Vector[EndClosed, s][[b]],
         {b, 1, 5}];
     , {type, {0, ei}}];
    , {s, {i, o}}];
```

Bounds for $N \geq 4$

Next, we bound the sum over all odd and even $N \geq 4$. We use the earlier computed decomposition of the vectors P and P^i in term of eigenvectors of A. We use these eigenvectors and the geometric sum to compute the sum of the bounds, see Section 5.4 of (I). We begin with the bound on the absolute value.

```
In[3159]:= Do[
  Do[
   vl[j] = Abs[EigenVector[left, j, s]];
   vi[j] = Abs[EigenVector[iota, j, s]];

   e[j] = Abs[EigenValue[j, s]];
   evi[j] = Abs[EigenValue[j, s]];
   , {j, 1, 5}];
  Bound[Xi, EvenTail, s] =
   muOverMub[s] × Sum[e[j]^2/(1 - e[j]^2) vl[j].Vector[EndOpen, s], {j, 1, 5}];
  Bound[Xi, OddTail, s] =
   muOverMub[s] × Sum[e[j]^3/(1 - e[j]^2) vl[j].Vector[EndOpen, s], {j, 1, 5}];
  Bound[XiIota, EvenTail, s] =
   muOverMub[s] × Sum[evi[j]^2/(1 - evi[j]^2) vi[j].Vector[EndOpen, s], {j, 1, 5}];
  Bound[XiIota, OddTail, s] =
   muOverMub[s] × Sum[evi[j]^3/(1 - evi[j]^2) vi[j].Vector[EndOpen, s], {j, 1, 5}];
  , {s, {i, o}}]
```

Then, we compute the bound on the weighted coefficients:

```
In[3160]:= Do[
  Do[
   vl[j] = Abs[EigenVector[left, j, s]];
   vr[j] = Abs[EigenVector[right, j, s]];
   vi[j] = Abs[EigenVector[iota, j, s]];
   e[j] = Abs[EigenValue[j, s]];
   evi[j] = Abs[EigenValue[j, s]];
   , {j, 1, 5}];
  Bound[Xi, EvenTail, Delta, s] =
   Vector[DeltaStart, s].Sum[vr[j] * e[j]^2 (2/(1 - e[j]^2)^2 + 2/(1 - e[j]^2)), {j, 1, 5}] +
    Sum[(2/(1 - e[j]^2)^2 + 2/(1 - e[j]^2)) * vl[j] e[j]^2, {j, 1, 5}].Vector[DeltaEnd, s] +
    Sum[2/(1 - e[j]^2)^2 * vl[j], {j, 1, 5}].
     (Matrix[C, s].Matrix[A, s] + Matrix[A, s].Matrix[C, s]).
      Sum[2/(1 - e[j]^2) vr[j], {j, 1, 5}] +
```

```
    Sum[2/(1 - e[j]^2) * vl[j], {j, 1, 5}].
     (Matrix[C, s].Matrix[A, s] + Matrix[A, s].Matrix[C, s]).
     Sum[2/(1 - e[j]^2)^2 vr[j], {j, 1, 5}];
  Bound[Xi, OddTail, Delta, s] =
   Vector[DeltaStart, s].Sum[vl[j] * e[j]^3 (2/(1 - e[j]^2)^2 + 3/(1 - e[j]^2)), {j, 1, 5}] +
    Sum[(2/(1 - e[j]^2)^2 + 3/(1 - e[j]^2)) * vl[j] e[j]^3, {j, 1, 5}].Vector[DeltaEnd, s] +
    Vector[DeltaStart, s].Matrix[C, s].
     Sum[vr[j] * e[j]^2 (2/(1 - e[j]^2)^2 + 3/(1 - e[j]^2)), {j, 1, 5}] +
    Sum[(2/(1 - e[j]^2)^2 + 1/(1 - e[j]^2)) * vl[j], {j, 1, 5}].Matrix[A, s].
     (Matrix[C, s].Matrix[A, s] + Matrix[A, s].Matrix[C, s]).
     Sum[2/(1 - e[j]^2) vr[j], {j, 1, 5}] +
    Sum[2/(1 - e[j]^2) * vl[j], {j, 1, 5}].Matrix[A, s].
     (Matrix[C, s].Matrix[A, s] + Matrix[A, s].Matrix[C, s]).
     Sum[2/(1 - e[j]^2)^2 vr[j], {j, 1, 5}];
  Do[
    Bound[XiIota, EvenTail, Delta, type, s] =
     Vector[D, type, s].Sum[vr[j] * e[j]^2 (2/(1 - e[j]^2)^2 + 2/(1 - e[j]^2)), {j, 1, 5}] +
      Sum[(2/(1 - evi[j]^2)^2 + 2/(1 - evi[j]^2)) * vi[j] * evi[j]^2, {j, 1, 5}].
        Vector[DeltaEnd, s] + Sum[2/(1 - evi[j]^2)^2 * vi[j], {j, 1, 5}].
        (Matrix[C, s].Matrix[A, s] + Matrix[A, s].Matrix[C, s]).
        Sum[2/(1 - e[j]^2) vr[j], {j, 1, 5}] +
      Sum[2/(1 - evi[j]^2) * vi[j], {j, 1, 5}].
        (Matrix[C, s].Matrix[A, s] + Matrix[A, s].Matrix[C, s]).
        Sum[2/(1 - e[j]^2)^2 vr[j], {j, 1, 5}];
```

```
Bound[XiIota, OddTail, Delta, type, s] =
  Vector[D, type, s].Sum[vr[j] * e[j]^3 (2/(1 - e[j]^2)^2 + 3/(1 - e[j]^2)), {j, 1, 5}] +
   Sum[evi[j]^3 (2/(1 - evi[j]^2)^2 + 3/(1 - evi[j]^2)) * vi[j], {j, 1, 5}].
    Vector[DeltaEnd, s] + Vector[P1Iota, s].Matrix[C, s].
    Sum[vr[j] * e[j]^2 (2/(1 - e[j]^2)^2 + 3/(1 - e[j]^2)), {j, 1, 5}] +
   Sum[(2/(1 - evi[j]^2)^2 + 1/(1 - evi[j]^2)) * vi[j], {j, 1, 5}].Matrix[A, s].
    (Matrix[C, s].Matrix[A, s] + Matrix[A, s].Matrix[C, s]).
    Sum[2/(1 - e[j]^2) vr[j], {j, 1, 5}] +
   Sum[2/(1 - evi[j]^2) * vi[j], {j, 1, 5}].Matrix[A, s].
    (Matrix[C, s].Matrix[A, s] + Matrix[A, s].Matrix[C, s]).
    Sum[2/(1 - e[j]^2)^2 vr[j], {j, 1, 5}];
 , {type, {0, ei}}] ×
 Clear[vl, vr, vi, e, evi]
, {s, {i, o}}]
```

Lower bounds

The analysis of (I) requires a number of lower bounds on the coefficients. Implementing some possible lower bounds, we found that while they improve our concluded bounds, using them did not allow us to lower the dimension in which we can obtain our results. So we decided to omit these lower bounds and their implications from our publication.

```
In[3161]:= Do[
   Bound[Pi, alpha, lower, 0, s] = 0;
   Bound[Psi, lower, 0, s] = 0;
   Bound[Pi, 1, Lower, s] = 0;
   , {s, {i, o}}];
```

Bounds for differences

To the bounds stated in Lemma 5.5 of (II). For these bounds we also decided to omit a number of terms, that we could have subtracted, as they did not allow us to prove the result in lower dimensions.

```
In[3162]:= Do[
  Bound[Xi, alpha, OneMinusZero, AtZero, s] = muOverMub[s] × (Bound[Loop, 4, s]);
  Bound[Xi, alpha, ZeroMinusOne, AtZero, s] = 0;

  Bound[Xi, alpha, OneMinusZero, AtEi, s] =
   1/(2 d) muOverMub[s] × (Bound[Bubble, 3, s] + 2 Bound[Loop, 4, s]);
  Bound[Xi, alpha, ZeroMinusOne, AtEi, s] = 0;
  Bound[Psi, alphaI, OneMinusZero, AroundEi, s] =
   2 gjz[s]^2 (Bound[tG, twoiNotusingi, 4, s] + (2 d - 2) Bound[tG, ikNotUsingi, 2, s]) +
    (2 d - 2) gjz[s]^2 (2 d gjz[s])^2 VarGamma2[s] × K[2, 2, {2}] +
    gjz[s]^2 (2 d gjz[s])^4 VarGamma2[s] × K[2, 4, {0, 1}];

  Bound[Psi, alphaI, ZeroMinusOne, AroundEi, s] =
   2 gjz[s]^2 (Bound[tG, twoiNotusingi, 4, s] + (2 d - 2) Bound[tG, ikNotUsingi, 2, s]);

  Bound[Psi, alphaII, OneMinusZero, AroundZero, s] =
   (2 d - 1) Bound[Xi, alpha, OneMinusZero, AtEi, s];

  Bound[Psi, alphaII, ZeroMinusOne, AroundZero, s] =
   (2 d - 1) Bound[Xi, alpha, ZeroMinusOne, AtEi, s];
  , {s, {i, o}}];
```

Summing the bounds

We compute the sum over all odd/even *N*:

In[3163]:=
```
Do[
 Bound[Xi, Even, s] = Sum[Bound[Xi, t, s], {t, {0, 2, EvenTail}}];
 Bound[Xi, Odd, s] = Sum[Bound[Xi, t, s], {t, {1, 3, OddTail}}];
 Bound[Xi, Absolut, s] = Sum[Bound[Xi, t, s], {t, {Odd , Even}}];

 Bound[Xi, Even, Delta, s] = Sum[Bound[Xi, t, Delta, s], {t, {0, 2, EvenTail}}];
 Bound[Xi, Odd, Delta, s] = Sum[Bound[Xi, t, Delta, s], {t, {1, 3, OddTail}}];
 Bound[Xi, Absolut, Delta, s] = Sum[Bound[Xi, t, Delta, s], {t, {Odd , Even}}];


 Bound[XiIota, Even, s] = Sum[Bound[XiIota, t, s], {t, {0, 2, EvenTail}}];
 Bound[XiIota, Odd, s] = Sum[Bound[XiIota, t, s], {t, {1, 3, OddTail}}];
 Bound[XiIota, Absolut, s] = Sum[Bound[XiIota, t, s], {t, {Odd , Even}}];

 Do[
  Bound[XiIota, Even, Delta, type, s] =
   Sum[Bound[XiIota, t, Delta, type, s], {t, {0, EvenTail}}];
  Bound[XiIota, Odd, Delta, type, s] =
   Sum[Bound[XiIota, t, Delta, type, s], {t, {1, OddTail}}];
  Bound[XiIota, Absolut, Delta, type, s] =
   Sum[Bound[XiIota, t, Delta, type, s], {t, {Odd, Even}}];
  , {type, {ei, 0}}];

 Clear[type, t];
 , {s, {i, o}}]
```

Bound on the simplified rewrite

In the preceding section we have computed all bounds required by Assumption 4.3 of (I). We use the methods provided in the General.nb-Notebook to compute the bounds on the simplified rewrite, as derived in Appendix D of (I).

In[3164]:=
```
Do[
 mu[s] = gjz[s];
 mub[s] = gz[s];
 mumin[s] = 1/(2 d - 1);

 beta[CPhi, Lower, s] =
  betaCPhiLow[d, mu[s], Bound[Xi, alpha, OneMinusZero, AtZero, s],
   Bound[XiIota, alphaI, 0, Atei, s]];
 beta[CPhi, Upper, s] =
  betaCPhiUp[d, mu[s], Bound[Xi, alpha, ZeroMinusOne, AtZero, s],
   Bound[XiIota, alphaII, 0, AtZero, s]];

 beta[af, Lower, s] = betaAfLow[d, mumin[s], mu[s],
   Bound[Psi, alphaI, OneMinusZero, AroundEi, s],
   Bound[Psi, alphaII, ZeroMinusOne, AroundZero, s], Bound[Pi, alpha, 0, s]];
 beta[af, Upper, s] =
```

```
 betaAfUp[d, mu[s], Bound[Psi, alphaI, ZeroMinusOne, AroundEi, s],
  Bound[Psi, alphaII, OneMinusZero, AroundZero, s],
  Bound[Pi, alpha, lower, 0, s]];

beta[ap, s] = betaap[d, mu[s], Bound[Xi, alpha, OneMinusZero, AtEi, s],
  Bound[Xi, alpha, ZeroMinusOne, AtEi, s],
  Bound[XiIota, alphaI, 0, SumAroundei, s],
  Bound[XiIota, alphaII, 0, SumAroundZero, s]];

beta[PiHat, s] = betaPiHat[d, mub[s], Bound[XiIota, Even, s],
  Bound[Pi, 1, Lower, s]];
beta[PsiHat, s] = betaPsiHatLower[d, mubOverMu[s], Bound[Xi, Odd, s],
  Bound[Psi, lower, 0, s]];


beta[Rf, s] = betaRF[d, mu[s], mub[s], mubOverMu[s], mub[s], Bound[Xi, Absolut, s],
  Bound[Xi, EvenTail, s] + Bound[Xi, OddTail, s], Bound[XiIota, Absolut, s],
  Sum[Bound[XiIota, t, s], {t, {Odd , EvenTail}}],
  Bound[Psi, RI, 0, s] + Bound[Psi, RI, 1, s],
  Bound[Psi, RII, 0, s] + Bound[Psi, RII, 1, s], Bound[Pi, R, 0, s]];


beta[Rp, s] = betaRp[d, mu[s], mubOverMu[s], mub[s], Bound[Xi, Absolut, s],
  Bound[Xi, R, 0, s] + Bound[Xi, R, 1, s],
  Bound[Xi, EvenTail, s] + Bound[Xi, OddTail, s], Bound[XiIota, Absolut, s],
  Sum[Bound[XiIota, t, s], {t, {Odd , EvenTail}}], Bound[XiIota, RI, 0, s],
  Bound[XiIota, RII, 0, s]];

beta[Rp, Delta, s] = betaRpDelta[d, mu[s], mubOverMu[s], mub[s],
  Bound[Xi, Absolut, s], Bound[Xi, Absolut, Delta, s],
  Bound[Xi, R, 0, s] + Bound[Xi, R, 1, s],
  Sum[Bound[Xi, t, Delta, s], {t, {OddTail , EvenTail}}],
  Bound[XiIota, Absolut, s], Bound[XiIota, Absolut, Delta, ei, s],
  Bound[XiIota, Absolut, Delta, 0, s],
  Sum[Bound[XiIota, t, Delta, ei, s], {t, {Odd, EvenTail}}],
  Sum[Bound[XiIota, t, Delta, 0, s], {t, {Odd, EvenTail}}],
  Bound[XiIota, RI, 0, Delta, ei, s], Bound[XiIota, RII, 0, Delta, 0, s]];


beta[Rf, abs, Delta, s] = betaRfDelta[d, mu[s], mubOverMu[s], mub[s],
  Bound[Xi, Absolut, s], Bound[Xi, Absolut, Delta, s],
  Bound[Xi, EvenTail, s] + Bound[Xi, OddTail, s],
  Sum[Bound[Xi, t, Delta, s], {t, {OddTail , EvenTail}}],
  Bound[Psi, RI, 0, Delta, s] + Bound[Psi, RI, 1, Delta, s],
  Bound[Psi, RII, 0, Delta, s] + Bound[Psi, RII, 1, Delta, s],
  Bound[XiIota, Absolut, s], Bound[XiIota, Absolut, Delta, ei, s],
  Bound[XiIota, Absolut, Delta, 0, s],
```

```
  Sum[Bound[XiIota, t, s], {t, {Odd, EvenTail}}],
  Sum[Bound[XiIota, t, Delta, ei, s], {t, {Odd, EvenTail}}],
  Bound[Pi, R, 0, Delta, eiek, s]];

beta[Rf, Lower, Delta, s] = betaRfDeltaLower[d, mu[s], mubOverMu[s], mub[s],
  Bound[Xi, Absolut, s], Bound[Xi, Odd, s], Bound[Xi, Even, s],
  Bound[Xi, Absolut, Delta, s], Bound[Xi, Odd, Delta, s],
  Bound[Xi, Even, Delta, s], Bound[Xi, OddTail, s], Bound[Xi, OddTail, Delta, s],
  Bound[Xi, EvenTail, Delta, s], Bound[Psi, RI, 1, Delta, s],
  Bound[Psi, RII, 0, Delta, s], Bound[XiIota, Absolut, s], Bound[XiIota, Odd, s],
  Bound[XiIota, Even, s], Bound[XiIota, Absolut, Delta, ei, s],
  Bound[XiIota, Odd, Delta, ei, s], Bound[XiIota, Even, Delta, ei, s],
  Bound[XiIota, Absolut, Delta, 0, s], Bound[XiIota, Odd, Delta, 0, s],
  Bound[XiIota, Odd, Delta, 0, s], Bound[XiIota, EvenTail, s],
  Bound[XiIota, EvenTail, Delta, ei, s], Bound[Pi, R, 0, Delta, eiek, s]];
, {s, {i, o}}]
```

Improvement of Bounds

In this section we implement the computations of Section 3 of (I) to verify whether we can conclude from $f_i(z) \leq \Gamma_i$ that $f_i(z) < \Gamma_i - \epsilon$. The sufficient condition for this to succeed is stated in Definition 2.9 of (I). We check the conditions one line at a time.

Technical conditions

All these conditions are necessary conditions. However, numerically they are next to trivial, in the sense that other conditions (most likely f_2 or f_3) will fail first.

```
In[3165]:= Do[
  TechCondition[I, s] = (beta[CPhi, Lower, s] - beta[ap, s] - beta[Rp, s]) > 0;
  (* Part of Assumption 2.7. of (I), stating that the nominator of Ĝz(k),
  being Φ̂(k), is positive *)
  TechCondition[II, s] = (beta[af, Lower, s] - beta[Rf, Lower, Delta, s]) > 0;
  (*Part of Assumption 2.7. of (I),
  neccesary to ensure that f2 is well defined *)
  TechCondition[III, s] = ((2 d - 1) mub[s] / (1 - mu[s]) Bound[XiIota, Absolut, s]) < 1;
  (* Condition (4.34) of (I),
  which is necessary to make the geometric series converge *)
  TechCondition[IV, s] = N[z[i]^4 (1 + (2 d - 2) z[i])^4] - Bound[tG, ikNotUsingi, 4, s]^2 >
    0;
  (* For lattice animals we verify that Ξ^(0),ι(eι+eκ)>Π^(0),ι,κ(eι+eκ),
  see (3.70) of (II).  Starting from (3.70) we derive a simpler numerical
   condition, that easier to verify. All details are explained below
   TODO Monoton?*)

  TechCondition[s] = TechCondition[I, s] && TechCondition[II, s] &&
    TechCondition[III, s] && TechCondition[IV, s]
  , {s, {i, o}}]
```

To verify condition IV we look at

$$\Sigma_{A:0,e_\iota+e_\kappa \in A}\, z^{|A|+1}\, 1_{A \Longleftrightarrow e_\iota+e_\kappa} \times (1_{e_\iota \in A} - 1_{e_\iota \neg\in A}) = \Sigma_{A:0,e_\iota+e_\kappa \in A}\, z^{|A|+1}\, 1_{A \Longleftrightarrow e_\iota+e_\kappa} \times (1_{e_\kappa \in A} - 1_{e_\iota \neg\in A})$$

For the right-most condition we see:

$$1_{e_\kappa \in A} - 1_{e_\iota \neg\in A} = 1_{e_\kappa, e_\iota \in A} + 1_{e_\kappa \in A} \times 1_{e_\iota \neg\in A} - (1_{e_\iota \neg\in A} \times 1_{e_\kappa \in A} + 1_{e_\iota, e_\kappa \neg\in A}) = 1_{e_\kappa, e_\iota \in A} - 1_{e_\iota, e_\kappa \neg\in A},$$

so that we need to verify

$$\Sigma_{A:0, e_\iota + e_\kappa \in \mathrm{A}}\, z^{|A|+1}\, 1_{A \Longleftrightarrow e_\iota + e_\kappa} \times (1_{e_\kappa, e_\iota \in A} - 1_{e_\iota, e_\kappa \neg\in A})$$

stating that there are more lattice animals (doubly connecting `0, e_ι + e_κ`) that use `e_ι` and `e_κ`, than not using both points. To prove this we use the lower bound:

$$\Sigma_{A:0, e_\iota, e_\kappa\, e_\iota + e_\kappa \in \mathrm{A}}\, z^{|A|}\, 1_{A \Longleftrightarrow e_\iota + e_\kappa} \geq \Sigma_A\, z^{|A|}\, 1_{(0, e_1) \in \mathrm{A}} \times 1_{(0, e_\kappa) \in \mathrm{A}} \times 1_{(e_\iota, e_\iota + e_\kappa) \in \mathrm{A}} \times 1_{(e_\kappa, e_\iota + e_\kappa) \in \mathrm{A}} \geq z^4 (1 + (2\,d - 2)\,z)^4$$

where z^4 corresponds to the 4 bonds of the small loop and $(1 + (2\,d - 2)\,z)^4$, four courner points,
with a possible additional bonds at each corner point. For the other term we use the upper bound

$$\Sigma_{A:0, e_\iota + e_\kappa \in \mathrm{A}}\, z^{|A|}\, 1_{A \Longleftrightarrow e_\iota + e_\kappa} \times 1_{e_\iota, e_\kappa \neg\in A} \leq \left(\Sigma_{A:0, e_\iota + e_\kappa \in \mathrm{A}}\, z^{|A|}\, 1_{A \Longleftrightarrow e_\iota + e_\kappa} \times 1_{e_\kappa \neg\in A} \times 1_{d_a(0, e_\iota + e_\kappa) > 2}\right)^2 \leq \text{Bound[tG, ikNotUsingi, 4, } o\text{]}$$

When the difference of this lower and this upper bound is positive than we have demonstrated the desired inequality (3.70) of (II).

Improvement of f_1

Following the bounds derived in Section 3.1 of (I):

In[3166]:=
```
Do[
  boundF1[part1, s] = mubOverMu[s] (1 + beta[PiHat, s]) / (1 - (2 d)/(2 d-1) beta[PsiHat, s]);
  boundF1[part2, s] = cmu (1 + beta[PiHat, s]) / (1 - (2 d)/(2 d-1) beta[PsiHat, s]);
  boundF1[s] = Max[boundF1[part1, s], boundF1[part2, s]];

  , {s, {i, o}}]
```

Improvement of f_2

Next we implement the bound derived in Section 3.2 of (I)

In[3167]:=
```
Do[
  boundF2[s] = (2 d - 1)/(2 d - 2) (beta[CPhi, Upper, s] + beta[ap, s] + beta[Rp, s]) / (beta[af, Lower, s] + beta[Rf, Lower, Delta, s]);
  , {s, {i, o}}]
```

Improvement of f_3

Here we start with some preparation and compute the value at z_I

In[3168]:=

```
(* The values to compare to *)
const[1] = GammaThreeClosed[1, 6];
const[2] = GammaThreeClosed[2, 6];
const[3] = GammaThree[1, 0];
const[4] = GammaThree[1, 1];
const[5] = GammaThree[1, 2];
const[6] = GammaThree[1, 3];
const[7] = GammaThree[2, 0];
const[8] = GammaThree[2, 1];
const[9] = GammaThree[2, 2];
const[10] = GammaThree[2, 3];

(* Inital bounds as in Section 3.3.3, obtained by pure SRW computation,
Methods provided in the General.nb *)
boundF3[1, i] = BoundFThreeInital[d, 1, 6, 1, {{0}} ];
boundF3[2, i] = BoundFThreeInital[d, 2, 6, 1, {{0}} ];

boundF3[3, i] = BoundFThreeInital[d, 1, 0, 1, {{1}, {2}, {0, 1}} ];
boundF3[4, i] = BoundFThreeInital[d, 1, 1, 1, {{1}, {2}, {0, 1}} ];
boundF3[5, i] = BoundFThreeInital[d, 1, 2, 1, {{1}, {2}, {0, 1}} ];
boundF3[6, i] = BoundFThreeInital[d, 1, 3, 1, {{1}, {2}, {0, 1}} ];

boundF3[7, i] = BoundFThreeInital[d, 2, 0, 1, {{1}, {2}, {0, 1}} ];
boundF3[8, i] = BoundFThreeInital[d, 2, 1, 1, {{1}, {2}, {0, 1}} ];
boundF3[9, i] = BoundFThreeInital[d, 2, 2, 1, {{1}, {2}, {0, 1}} ];
boundF3[10, i] = BoundFThreeInital[d, 2, 3, 1, {{1}, {2}, {0, 1}} ];
```

In[3188]:=

In[3189]:=
```
(* The bounds summarized in (3.80) *)
BoundFThreeBound[n_, l_, vs_] :=
  BoundFThree[d, n, l, vs, Gamma2, beta[CPhi, Upper, o], beta[af, Lower, o],
   beta[af, Upper, o], beta[ap, o], beta[Rp, o], beta[Rf, o], beta[Rf, abs, Delta, o],
   beta[Rp, Delta, o], 1/(beta[af, Lower, o] + beta[Rf, Lower, Delta, o])];

boundF3[1, o] := BoundFThreeBound[1, 6, {{0}}];
boundF3[2, o] = BoundFThreeBound[2, 6, {{0}}];

BoundFThreeOpen[n_, l_] := BoundFThreeBound[n, l, {{1}, {2}, {0, 1}}];

onstepContribution = BoundFThreeBound[1, 2, {{1}}] +
   1/(2 d) (2 × (2 d - 2) Bound[tG, {2}, 2, o] + 4 Bound[tG, {0, 1}, 2, o]);
boundF3[3, o] = Max[2 d gjz[o] onstepContribution + Bound[tG, {1}, 1, o],
   2 d gjz[o] × BoundFThreeBound[1, 1, {{2}}] + Bound[tG, {2}, 2, o],
   2 d gjz[o] × BoundFThreeBound[1, 1, {{0, 1}}] + Bound[tG, {0, 1}, 2, o],
   BoundFThreeBound[1, 0, {{3}, {1, 1}, {0, 0, 1}}]];
boundF3[4, o] = Max[onstepContribution, BoundFThreeBound[1, 1, {{2}, {0, 1}}]];
boundF3[5, o] = BoundFThreeOpen[1, 2];
boundF3[6, o] = BoundFThreeOpen[1, 3];

boundF3[7, o] =
  Max[2 d gjz[o] × (BoundFThreeBound[1, 1, {{1}}] + BoundFThreeBound[2, 1, {{1}}]) +
     Bound[tG, {1}, 2, o],
    2 d gjz[o] × (BoundFThreeBound[1, 1, {{2}}] + BoundFThreeBound[2, 1, {{2}}]) +
     Bound[tG, {2}, 2, o],
    2 d gjz[o] × (BoundFThreeBound[1, 1, {{0, 1}}] + BoundFThreeBound[2, 1, {{0, 1}}]) +
     + Bound[tG, {0, 1}, 2, o], BoundFThreeBound[2, 0, {{3}, {1, 1}, {0, 0, 1}}]];
boundF3[8, o] = BoundFThreeOpen[2, 1];
boundF3[9, o] = BoundFThreeOpen[2, 2];
boundF3[10, o] = BoundFThreeOpen[2, 3];


BoundsF3Table = Table[boundF3[j, s] / const[j], {s, {i, o}}, {j, 1, 10}];
boundF3[i] = Ceiling[Max[BoundsF3Table[[1]]], 10^-9];
boundF3[o] = Max[BoundsF3Table[[2]]];
```

Results

Preparation of output

In[3204]:=
```
Do[
  SuccesF[1, s] = boundF1[s] < Gamma1;
  SuccesF[2, s] = boundF2[s] < Gamma2;
  SuccesF[3, s] = boundF3[s] < Gamma3;
  Succes[s] = SuccesF[1, s] && SuccesF[2, s] && SuccesF[3, s] && TechCondition[s];
  , {s, {i, o}}];
Succes[overall] = Succes[i] && Succes[o];
```

In[3206]:=
```
overAllStatement = "The statement that the bootstrap was successful is "
   If[Succes[overall], Style[TextString[Succes[overall]], Bold, Green],
    Style[TextString[Succes[overall]], Bold, Red]];
```

In[3207]:=
```
bubbles = {Graphics[{Green, Disk[{0, 0}, 0.8]}, ImageSize → 30],
   Graphics[{Red, Disk[{0, 0}, 0.8]}, ImageSize → 40]};
```

```
Do[
 CoefficientboundsTable[s] =
  {{Quantity, "Ξ^Zero", "Ξ^One", "Ξ^Two", "Ξ^Three", "Ξ^Even,>3", "Ξ^Odd,>3"},
   {Text[Bound for Ξ̂], Bound[Xi, 0, s], Bound[Xi, 1, s] , Bound[Xi, 2, s],
    Bound[Xi, 3, s] , Bound[Xi, EvenTail, s], Bound[Xi, OddTail, s]},
   {Text[Bound for Ξ̂^ι], Bound[XiIota, 0, s], Bound[XiIota, 1, s] ,
    Bound[XiIota, 2, s], Bound[XiIota, 3, s] , Bound[XiIota, EvenTail, s],
    Bound[XiIota, OddTail, s]},
   {Text[Ξ̂ "|x|_2^2" ], Bound[Xi, 0, Delta, s], Bound[Xi, 1, Delta, s],
    Bound[Xi, 2, Delta, s], Bound[Xi, 3, Delta, s], Bound[Xi, EvenTail, Delta, s],
    Bound[Xi, OddTail, Delta, s]},
   {Text[Ξ^ι "|x-e_ι|_2^2"] , Bound[XiIota, 0, Delta, ei, s],
    Bound[XiIota, 1, Delta, ei, s]  , Bound[XiIota, 2, Delta, ei, s],
    Bound[XiIota, 3, Delta, ei, s], Bound[XiIota, EvenTail, Delta, ei, s],
    Bound[XiIota, OddTail, Delta, ei, s]},
   {Text[Ξ^ι "|x|_2^2"] , Bound[XiIota, 0, Delta, 0, s], Bound[XiIota, 1, Delta, 0, s] ,
    Bound[XiIota, 2, Delta, 0, s], Bound[XiIota, 3, Delta, 0, s],
    Bound[XiIota, EvenTail, Delta, 0, s], Bound[XiIota, OddTail, Delta, 0, s]}};
 TableCoefficients[s] =
  Labeled[Grid[CoefficientboundsTable[s], Alignment → {Center}, Frame → True,
     Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
     Background → {{None}, {GrayLevel[0.9]}, {None}}],
    Style["Bounds on the coefficients in dimension in " <> TextString[d], Bold],
    Top] // Text;


 CoefficientSplitsBoundsI[s] =
  {{Quantity, "Ξ_R^Zero", "Ψ_RI^Zero", "Ψ_RII^Zero", "Ξ_R^One", "Ψ_RI^One", "Ψ_RII^One"},
   {Text[Abs Bound], Bound[Xi, R, 0, s], Bound[Psi, RI, 0, s],
```

```
    Bound[Psi, RII, 0, s], Bound[Xi, R, 1, s] , Bound[Psi, RI, 1, s],
    Bound[Psi, RII, 1, s]}, {Text[Ξ̂ "|x|_2^2" ], Bound[Xi, R, 0, Delta, s],
    Bound[Psi, RI, 0, Delta, s], Bound[Psi, RII, 0, Delta, s],
    Bound[Xi, R, 1, Delta, s] , Bound[Psi, RI, 1, Delta, s],
    Bound[Psi, RII, 1, Delta, s]}};
TableCoefficientSplitI[s] =
 Labeled[Grid[CoefficientSplitsBoundsI[s], Alignment → {Center},
    Frame → True, Dividers → {{2 → True, -1 → True}, {2 → True}},
    ItemStyle → {1 → Bold, 1 → Bold},
    Background → {{None}, {GrayLevel[0.9]}, {None}}],
   Style["Bounds on the remainder term of the split of Ξ and Ψ in d=" <>
     TextString[d], Bold], Top] // Text;

CoefficientSplitsBoundsII[s] =
 {{Quantity, " Ξ̂^zero_alpha(0)-Ξ̂^One_alpha(0)", " Ξ̂^zero_alpha(e1)-Ξ̂^One_alpha(e1)", "∑Ψ^Zero_alphaI-Ψ^One_alphaI",
    "∑Ψ^Zero_alphaII-Ψ^One_alphaII"},
  {Text[Lower Bound], Bound[Xi, alpha, OneMinusZero, AtZero, s],
   Bound[Xi, alpha, OneMinusZero, AtEi, s],
   Bound[Psi, alphaI, OneMinusZero, AroundEi, s],
   Bound[Psi, alphaII, OneMinusZero, AroundZero, s]},
  {Text[Upper Bound ], Bound[Xi, alpha, ZeroMinusOne, AtZero, s],
   Bound[Xi, alpha, ZeroMinusOne, AtEi, s],
   Bound[Psi, alphaI, ZeroMinusOne, AroundEi, s],
   Bound[Psi, alphaII, ZeroMinusOne, AroundZero, s]}};
TableCoefficientSplitII[s] =
 Labeled[Grid[CoefficientSplitsBoundsII[s], Alignment → {Center},
    Frame → True, Dividers → {{2 → True, 3 → True, 4 → True, 5 → True, -1 → True},
      {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
    Background → {{None}, {GrayLevel[0.9]}, {None}}],
   Style["Bounds on the difference of explicit terms in dimension " <>
     TextString[d], Bold], Top] // Text;

CoefficientSplitsBoundsIII[s] =
 {{Quantity, " Ξ̂^ι", "|x-e_ι|_2^2 Ξ̂^ι", " Ξ̂^ι_alphaI(e_ι)", "∑ Ξ̂^ι_alphaI", " Ξ̂^ι_RI",
    "|x-e_ι|_2^2 Ξ̂^ι_RI", "|x|_2^2 Ξ̂^ι", " Ξ̂^ι_alphaII(0)", "∑ Ξ̂^ι_alphaII", "Ξ̂^ι_RII", "|x|_2^2 Ξ̂^ι_RII"},
  {, Bound[XiIota, 0, s], Bound[XiIota, 0, Delta, ei, s],
   Bound[XiIota, alphaI, 0, Atei, s], Bound[XiIota, alphaI, 0, SumAroundei, s],
   Bound[XiIota, RI, 0, s], Bound[XiIota, RI, 0, Delta, ei, s],
   Bound[XiIota, 0, Delta, 0, s], Bound[XiIota, alphaII, 0, AtZero, s],
   Bound[XiIota, alphaII, 0, SumAroundZero, s], Bound[XiIota, RII, 0, s],
   Bound[XiIota, RII, 0, Delta, 0, s]}};
TableCoefficientSplitIII[s] =
 Labeled[Grid[CoefficientSplitsBoundsIII[s], Alignment → {Center},
    Frame → True, Dividers → {{2 → True, 5 → True, 8 → True, -1 → True}, {2 → True}},
    ItemStyle → {1 → Bold, 1 → Bold},
    Background → {{None}, {GrayLevel[0.9]}, {None}}],
```

```
    Style["Bounds on split of the coefficient Ξ^(0),ι in dimension " <>
      TextString[d], Bold], Top] // Text;
 , {s, {i, o}}]

SimpleNotationBoundsPart1 =
  {{Quantity, "aF-Lower", "aF-Upper", "|aΦ|", "cΦ-Lower", "cΦ-Upper", Π, Ψ},
   {"Bound i" , beta[af, Lower, i], beta[af, Upper, i], beta[ap, i],
    beta[CPhi, Lower, i], beta[CPhi, Upper, i], beta[PiHat, i], beta[PsiHat, i]},
   {"Bound o" , beta[af, Lower, o], beta[af, Upper, o], beta[ap, o],
    beta[CPhi, Lower, o], beta[CPhi, Upper, o], beta[PiHat, o], beta[PsiHat, o]}};

TableSimpleNotation1 =
  Labeled[Grid[SimpleNotationBoundsPart1, Alignment → {Center}, Frame → True,
     Dividers → {{2 → True, -1 → True}, {2 → True}},
     ItemStyle → {1 → Bold, 1 → Bold},
     Background → {{None}, {GrayLevel[0.9]}, {None}}],
    Style["Bounds simplified rewrite in dimension " <> TextString[d], Bold], Top] //
   Text;

SimpleNotationBoundsPart2 =
  {{Quantity, "|RF|", "|RΦ|", "x2²RF-lower", "|x2²RF|", "|x2²RΦ|"},
   {"Bound i", beta[Rf, i], beta[Rp, i], beta[Rf, Lower, Delta, i],
    beta[Rf, abs, Delta, i], beta[Rp, Delta, i]},
   {"Bound o", beta[Rf, o], beta[Rp, o], beta[Rf, Lower, Delta, o],
    beta[Rf, abs, Delta, o], beta[Rp, Delta, o]}};
TableSimpleNotation2 =
  Labeled[Grid[SimpleNotationBoundsPart2, Alignment → {Center}, Frame → True,
     Dividers → {{2 → True, 4 → True, -1 → True}, {2 → True}},
     ItemStyle → {1 → Bold, 1 → Bold},
     Background → {{None}, {GrayLevel[0.9]}, {None}}],
    Style["Bounds on remainder of rewrite in dimension " <> TextString[d], Bold],
    Top] // Text;

ContentCheckf1f2 = {{Bounds, "f1(zI)", "f2(zI)", "f3(zI)", "f1(z)", "f2(z)", "f3(z)"},
   {"Γi", NumberForm[N[Gamma1], 10], NumberForm[N[Gamma2], 10],
    NumberForm[N[Gamma3], 10], NumberForm[N[Gamma1], 10], NumberForm[N[Gamma2], 10],
    NumberForm[N[Gamma3], 10] },
   {Bounds, NumberForm[boundF1[i], 10], NumberForm[boundF2[i], 10],
    NumberForm[N[boundF3[i]], 10], NumberForm[boundF1[o], 10],
    NumberForm[boundF2[o], 10], NumberForm[boundF3[o], 10]}, { "check",
    If[SuccesF[1, i], bubbles[[1]], bubbles[[2]]],
    If[SuccesF[2, i], bubbles[[1]], bubbles[[2]]],
    If[SuccesF[3, i], bubbles[[1]], bubbles[[2]]],
    If[SuccesF[1, o], bubbles[[1]], bubbles[[2]]],
    If[SuccesF[2, o], bubbles[[1]], bubbles[[2]]],
    If[SuccesF[3, o], bubbles[[1]], bubbles[[2]]]}};
```

```
TableCheckf1f2 =
  Labeled[Grid[ContentCheckf1f2, Alignment → {Center}, Frame → True,
     Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
     Background → {{None}, {GrayLevel[0.9]}, {None}}],
    Style["Bounds on the LA-bootstrap functions in dimension " <> TextString[d],
     Bold], Top] // Text;

Do[
 tableCheckf3Bubble[s] =
  {{Bounds, "F3-1,6,{0}", "F3-1,0,x≠0", "F3-1,1,x≠0", "F3-1,2,x≠0", "F3-1,3,x≠0"},
   {"Assumed bound", NumberForm[N[const[1] * Gamma3], 10],
    NumberForm[N[const[3] * Gamma3], 10], NumberForm[N[const[4] * Gamma3], 10],
    NumberForm[N[const[5] * Gamma3], 10], NumberForm[N[const[6] * Gamma3], 10] },
   {"Concluded bound", boundF3[1, s], boundF3[3, s], boundF3[4, s],
    boundF3[5, s], boundF3[6, s]}, { "check",
    If[boundF3[1, s] < const[1] * Gamma3, bubbles⟦1⟧, bubbles⟦2⟧],
    If[boundF3[3, s] < const[3] * Gamma3, bubbles⟦1⟧, bubbles⟦2⟧],
    If[boundF3[4, s] < const[4] * Gamma3, bubbles⟦1⟧, bubbles⟦2⟧],
    If[boundF3[5, s] < const[5] * Gamma3, bubbles⟦1⟧, bubbles⟦2⟧],
    If[boundF3[6, s] < const[6] * Gamma3, bubbles⟦1⟧, bubbles⟦2⟧]}};

 tableCheckf3Triangle[s] =
  {{Bounds, "F3-2,6,{0}", "F3-2,0,x≠0", "F3-2,1,x≠0", "F3-2,2,x≠0", "F3-2,3,x≠0"},
   {"Assumed bound", NumberForm[N[const[2] * Gamma3], 10] ,
    NumberForm[N[const[7] * Gamma3], 10], NumberForm[N[const[8] * Gamma3], 10],
    NumberForm[N[const[9] * Gamma3], 10], NumberForm[N[const[10] * Gamma3], 10] },
   {"Concluded bound", boundF3[2, s], boundF3[7, s], boundF3[8, s],
    boundF3[9, s], boundF3[10, s]}, { "check",
    If[boundF3[2, s] < const[2] * Gamma3, bubbles⟦1⟧, bubbles⟦2⟧],
    If[boundF3[7, s] < const[7] * Gamma3, bubbles⟦1⟧, bubbles⟦2⟧],
    If[boundF3[8, s] < const[8] * Gamma3, bubbles⟦1⟧, bubbles⟦2⟧],
    If[boundF3[9, s] < const[9] * Gamma3, bubbles⟦1⟧, bubbles⟦2⟧],
    If[boundF3[10, s] < const[10] * Gamma3, bubbles⟦1⟧, bubbles⟦2⟧]}};

 , {s, {i, o}}]

TableCheckf3Bubble =
  Labeled[Grid[tableCheckf3Bubble[o], Alignment → {Center}, Frame → True,
     Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
     Background → {{None}, {GrayLevel[0.9]}, {None}}],
    Style["Bounds on the LA-weighted bubble in dimension " <> TextString[d], Bold],
    Top] // Text;

TableCheckf3Triangle =
  Labeled[Grid[tableCheckf3Triangle[o], Alignment → {Center}, Frame → True,
     Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
     Background → {{None}, {GrayLevel[0.9]}, {None}}],
    Style["Bounds on the LA-weighted triangle in dimension " <> TextString[d],
     Bold], Top] // Text;
```

Simple output

In[3218]:=
```
overAllStatement
```

Out[3218]= The statement that the bootstrap was successful is **True**

If this succeeds, then the infrared bound Theorem 2.10 of (I) or Theorem 1.1 of (II) holds with

In[3219]:=
```
"Ĝz(k)[1-D̂(k)] ≤ " <> TextString[Ceiling[(2 d - 2)/(2 d - 1) Gamma2, 0.001]]
"Ĝz(k)[1-D̂(k)] ≤ " <> TextString[Ceiling[g[o] (2 d - 2)/(2 d - 1) Gamma2, 0.001]] <>
 " = A2(d)"
"A1(d) = " <>
 TextString[
  Ceiling[g[o] × (beta[CPhi, Upper, o] + beta[ap, o] + beta[Rp, o])
    Max[1/(beta[af, Lower, o] + beta[Rf, Lower, Delta, o]),
     1/(beta[CPhi, Lower, o] - beta[ap, o] - beta[Rp, o])], 0.001]]
```

Out[3219]= $\hat{G}_z(k)\,[1-\hat{D}(k)] \leq 1.207$

Out[3220]= $\hat{\overline{G}}_z(k)\,[1-\hat{D}(k)] \leq 3.587 = A_2(d)$

Out[3221]= $A_1(d) = 3.587$

Further, we have proven that g_{z_c} and $g_{z_c} z_c$, respectively are smaller than:

In[3222]:=
```
Print["g_{z_c} ≤ ", Ceiling[Exp[1] Gamma1, 0.0001]]
Print["(2d-1) z_c g_{z_c} ≤ ", Ceiling[Gamma1, 0.0001]]
```

$g_{z_c} \leq 2.9721$

$(2d-1)\, z_c g_{z_c} \leq 1.0934$

Bound on the bootstrap functions

In[3224]:=
```
TableCheckf1f2
TableCheckf3Bubble
TableCheckf3Triangle
```

Out[3224]=

Bounds on the LA–bootstrap functions in dimension 17

Bounds	$f_1(z_l)$	$f_2(z_l)$	$f_3(z_l)$	$f_1(z)$	$f_2(z)$	$f_3(z)$
Γ_i	1.09335749	1.24459253	1.	1.09335749	1.24459253	1.
Bounds	1.081859344	1.159818969	0.249263135	1.093357479	1.244592519	0.9999999654
check						

Out[3225]=

Bounds on the LA–weighted bubble in dimension 17

Bounds	F3–1,6,{0}	F3–1,0,x≠0	F3–1,1,x≠0	F3–1,2,x≠0	F3–1,3,x≠0
Assumed bound	0.0017725	0.15664574	0.0411822	0.0192609	0.00771789
Concluded bound	0.00177249	0.156646	0.0411821	0.0192609	0.00771786
check					

Out[3226]=

Bounds on the LA–weighted triangle in dimension 17

Bounds	F3–2,6,{0}	F3–2,0,x≠0	F3–2,1,x≠0	F3–2,2,x≠0	F3–2,3,x≠0
Assumed bound	0.00520848	0.2532804	0.11245001	0.04589325	0.02025117
Concluded bound	0.00520846	0.25328	0.11245	0.0458932	0.0202511
check					

Bounds of the coefficients

In[3227]:=
```
TableCoefficients[o]
TableSimpleNotation1
TableSimpleNotation2
TableCoefficientSplitI[o]
TableCoefficientSplitII[o]
TableCoefficientSplitIII[o]
```

Bounds on the coefficients in dimension in 17

Out[3227]=

Quantity	Ξ^{Zero}	Ξ^{One}	Ξ^{Two}	Ξ^{Three}	$\Xi^{Even,>3}$	$\Xi^{Odd,>3}$
Bound for $\hat{\Xi}$	0.00161798	0.01792	0.00127632	0.0000904419	7.43272×10^{-6}	5.64172×10^{-7}
Bound for $\hat{\Xi}'$	0.0334643	0.00158016	0.000111219	7.88181×10^{-6}	6.47376×10^{-7}	4.91405×10^{-8}
$\lvert x\rvert_2^2\,\hat{\Xi}$	0.0041594	0.0429262	0.0590374	0.00993962	0.00278227	0.000689626
$\lvert x-e_\iota\rvert_2^2\,\Xi'$	0.000547384	0.0136256	0.00719464	0.00106675	0.000264443	0.0000281703
$\lvert x\rvert_2^2\,\Xi'$	0.000288095	0.00483864	0.00544581	0.000894218	0.000242562	0.0000260954

Bounds simplified rewrite in dimension 17

Out[3228]=

Quantity	a_F–Lower	a_F–Upper	$\lvert a_\Phi\rvert$	c_Φ–Lower	c_Φ–Upper	Π	Ψ
Bound i	1.03035	1.03146	0.0372438	0.997705	1.000978939065-374834549622-6142053	0.033056	0.012912457884-847716287129-770
Bound o	1.02916	1.09161	0.0429384	0.996878	1.00117	0.0378232	0.0186109

Bounds on remainder of rewrite in dimension 17

Out[3229]=

Quantity	$\lvert R_F\rvert$	$\lvert R_\Phi\rvert$	$x_2^2 R_F$–lower	$\lvert x_2^2 R_F\rvert$	$\lvert x_2^2 R_\Phi\rvert$
Bound i	0.121141	0.00996421	−0.0983587	0.112454	0.013983
Bound o	0.152766	0.0151419	−0.151486	0.190417	0.036883

Bounds on the remainder term of the split of Ξ and Ψ in d=17

Out[3230]=

Quantity	Ξ_R^{Zero}	Ψ_{RI}^{Zero}	Ψ_{RII}^{Zero}	Ξ_R^{One}	Ψ_{RI}^{One}	Ψ_{RII}^{One}
Abs Bound	0.00075434	0.00302292	0.00075434	0.00986543	0.016235	0.00986543
$\lvert x\rvert_2^2\,\hat{\Xi}$	0.003267	0.00718232	0.003267	0.0429262	0.0591612	0.0429262

Bounds on the difference of explicit terms in dimension 17

Out[3231]=

Quantity	$\hat{\Xi}_{alpha}^{zero}(0)-\hat{\Xi}_{alpha}^{One}(0)$	$\hat{\Xi}_{alpha}^{zero}(e_1)-\hat{\Xi}_{alpha}^{One}(e_1)$	$\Sigma\Psi_{alphaI}^{Zero}-\Psi_{alphaI}^{One}$	$\Sigma\Psi_{alphaII}^{Zero}-\Psi_{alphaII}^{One}$
Bound Lower	0.00135668	0.000192168	0.00025527	0.00634156
Bound Upper	0	0	0.0000815567	0

Bounds on split of the coefficient $\Xi^{(0),\prime}$ in dimension 17

Out[3232]=

Quantity	$\hat{\Xi}'$	$\lvert x-e_I \rvert_2^2 \hat{\Xi}'$	$\hat{\Xi}'_{alphaI}(e_I)$	$\Sigma \hat{\Xi}'_{alphaI}$	$\hat{\Xi}'_{RI}$	$\lvert x-e_I \rvert_2^2 \hat{\Xi}'_{RI}$	$\lvert x \rvert_2^2 \hat{\Xi}'$	$\hat{\Xi}'_{alphaII}(0)$	$\Sigma \hat{\Xi}'_{alphaII}$	$\hat{\Xi}'_{RII}$	$\lvert x \rvert_2^2 \hat{\Xi}'_{RII}$
	0.033464-3	0.000547-384	0.0016-1798	0.0333-59	0.000127-589	0.000541-713	0.000288-095	0.0333-085	0.00018379-1	0.000092424-6	0.000241-799

Algorithm to find good values for the constants

The follow is a semi-automate procedure to find appropriate values for the constants Γ_i and c_i.

How to use it: Initially, we guess a good value for the constants and make a first computation. Then, we deactivate the declaration of the constants at the very beginning of this document. Then we recompile the entire document multiple times (Menu Evaluate>>Evaluate notebook) and hope that the algorithm below converges to a fixed-point for the parameters.

The idea to use the previously concluded bounds(+ ε) as new initial values for the Γ's and constants. Then, recompile and show that we can conclude the same bounds starting from these values. As initial value we recommend to either use the values of the bounds at z_I, denoted by s=i, which are independent of the constants Γ_i and c_i. Another reasonable choice would be to start with values that work in a slightly higher dimension.

In[3233]:=

```
Gamma1 = Ceiling[Max[boundF1[i], boundF1[o]], 10^-8] + 10^-8;
Gamma2 = Ceiling[Max[boundF2[i], boundF2[o]], 10^-8] + 10^-8;


GammaThreeClosed[1, 6] = Ceiling[Max[boundF3[1, i], boundF3[1, o]], 10^-8] + 10^-8;
GammaThreeClosed[2, 6] = Ceiling[Max[boundF3[2, i], boundF3[2, o]], 10^-8] + 10^-8;
GammaThree[1, 0] = N[Ceiling[Max[boundF3[3, i], boundF3[3, o]], 10^-8] + 10^-8];
GammaThree[1, 1] = Ceiling[Max[boundF3[4, i], boundF3[4, o]], 10^-8] + 10^-8;
GammaThree[1, 2] = Ceiling[Max[boundF3[5, i], boundF3[5, o]], 10^-8] + 10^-8;
GammaThree[1, 3] = Ceiling[Max[boundF3[6, i], boundF3[6, o]], 10^-8] + 10^-8;
GammaThree[2, 0] = Ceiling[Max[boundF3[7, i], boundF3[7, o]], 10^-8] + 10^-8;
GammaThree[2, 1] = Ceiling[Max[boundF3[8, i], boundF3[8, o]], 10^-8] + 10^-8;
GammaThree[2, 2] = Ceiling[Max[boundF3[9, i], boundF3[9, o]], 10^-8] + 10^-8;
GammaThree[2, 3] = Ceiling[Max[boundF3[10, i], boundF3[10, o]], 10^-8] + 10^-8;

cmu = mubOverMu[o] + 0.000001;
```