Computation for the lace expansion for lattice trees

Analysis of Section 3.5-3.6

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Abstract

In this file we performs the numerical part of the analysis of the non-backtracking lace expansion for lattice trees. All references in this version of the notebook will be to the PhD thesis of the author.

This file is accomanied by another notebook -SRW_Computations- where an number of simple random walks are computed. The user should first open that file, choose a dimension and execute all lines of the file. Then he is expects to choose constance Γ_i in this file. After choosing these quantities the used should select the menu item Evaluate-> Evaluate Notebook. In a table at the end of this document the result of the computations are shown. There it can be see whether the bootstrap with the given parameters and therefore the analysis was succesful. The computation of the -SRW_Computations- file are independent of the values Γ_i, so that the need to compute the SRW-integral once when we start the program and whenever we change the dimension.

We compute bound on the two-point function and repulsive diagrams (Section 5.1.2). We use these bounds to compute the Diagramatic bounds derived in Section 4.3. and compute the bounds used for the Analysis in Section 3.5.

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Input

We try to perform the bootstrap for the following values of Γ_i. So that the values of f_1, f_2,$\bar{f}_{3,n,l}$ and $\bar{f}_{4,n,l}$ ars small then:

In[5036]:=
```
Gamma1 = 1.0546357;
Gamma2 = 1.1433;

GammaFour[1, 4] = 0.003234;
GammaFour[2, 4] = 0.0067444;

GammaThree[1, 0] = 0.1367;
GammaThree[1, 1] = 0.026228;
GammaThree[1, 2] = 0.010843;
GammaThree[1, 3] = 0.003234;
GammaThree[2, 0] = 0.20456;
GammaThree[2, 1] = 0.04752;
GammaThree[2, 2] = 0.01965;
GammaThree[2, 3] = .00675;
```

The bootstraps succeeds in dimension 20 with the constants (* deactivated*)

In[5048]:=
```
(*Gamma1=1.0546357;
Gamma2=1.1433;

GammaFour[1,4]=0.003234;
GammaFour[2,4]=0.0067444;

GammaThree[1,0]=0.1367;
GammaThree[1,1]=0.026228;
GammaThree[1,2]=0.010843;
GammaThree[1,3]=0.003234;
GammaThree[2,0]=0.20456;
GammaThree[2,1]=0.04752;
GammaThree[2,2]=0.01965;
GammaThree[2,3]=.00675;*)
```

Bound on the two-point function and on repulsive diagrams

Definition of Constants

We define the constants for two setting s: we use s=i for bound on $z = z_I$ and s=o for bound on $z \epsilon (z_I, z_c)$: Further we use the following relations,

$$\begin{aligned} z_I &= \frac{1}{(2d-1)e} \\ g_{z_I} &\leq e + \frac{e-1}{2d-1} \\ g^i_{z_I} &\leq 1 + (g_{z_I} - 1)\frac{2d-1}{2d} \leq e \\ G_z(x) &\leq g^i_{z_I} B_{z_I g^i_{z_I}}(x) \leq g^i_{z_I} \frac{2d-2}{2d-1} C(x) \\ \tilde{G}_z(x) &\leq B_{z_I g^i_{z_I}}(x) \leq \frac{2d-2}{2d-1} C(x). \end{aligned} \tag{1}$$

For the other z ϵ (z_I, z_c) we know that

$$\begin{aligned} 2dz g^i_z &< 2d g_z z < \Gamma_1 \\ g_z &< e\,\Gamma_1 \text{ for all } z \geq \frac{1}{(2d-1)e} \\ g^i_z &< 1 + (g_z - 1)\frac{2d-1}{2d} \end{aligned} \tag{2}$$

We implement the following basic quantities for status i at z_I and status o for z in (z_I, z_c), which will allows us to implement the bound for both (s=i,o) at the same time:

In[5049]:=
```
g[i] = Exp[1] + (Exp[1] - 1)/(2 d - 1);
g[o] = Exp[1] * Gamma1;
gj[i] = Exp[1];
gj[o] = 1 + (g[o] - 1) * (2 d - 1)/(2 d);
rho[i] = gj[i]/g[i];
rho[o] = gj[o]/g[o];


twodgz[i] = 2 d 1/((2 d - 1) Exp[1]) g[i];
twodgz[o] = (2 d)/(2 d - 1) Gamma1;
twodgjz[i] = (2 d)/(2 d - 1);
twodgjz[o] = (2 d)/(2 d - 1) Gamma1;
gjz[i] = 1/(2 d - 1);
gjz[o] = 1/(2 d - 1) Gamma1;

(*twodz[i]=2d z[i];
twodz[o]= 2d z[o];
*)
(*bound on the two-point function*)
VarGamma1[i] = (1 + (1 - Exp[-1])/(2 d - 1));
VarGamma1[o] = Gamma1;
VarGamma2[i] = rho[i] * (2 d - 2)/(2 d - 1); (*G_z(x) ≤g_z B_{zg_z}(x) ≤  g_z (2d-2)/(2d-1) C(x) *)
VarGamma2[o] = Gamma2 * (2 d - 2)/(2 d - 1); (*Ĝ_z(k)≤  ConstantGvsC Ĉ(k), follows from f2*)
VarGamma3[i] = 1; (*We bound a weighted line by replacing the tree two-
 point function with a normal on*)
VarGamma3[o] = Gamma3;(*Ĝ_z(k)≤  Ĉ(k)*)
Varc1[o] = c1; Varc2[o] = c2; Varc3[o] = c3; Varc4[o] = c4;
Varc1[i] = 0; Varc2[i] = 0.5; Varc3[i] = 0; Varc4[i] = 4;
```

Further, we define variables to save the number of short SAWs, as given in Section 5.1.3

In[5069]:=
```
c2ik = 2; (*c2(e1+e2)*)
c4ik = 4 (2 d - 3) + 2 (2 d - 4); (*c4(e1+e2)*)
c6ik = 16 + 84 (2 d - 4) + 36 (2 d - 4) (2 d - 6); (*c4(e1+e2)*)

c3i = (2 d - 2); (*c3(e1)*)
c5i = (3 (2 d - 2 ) + 4 (2 d - 2) (2 d - 4) ); (*c5(e1)*)
c7i = (14 (2 d - 2 ) + 62 (2 d - 2) (2 d - 4) + 27 (2 d - 2) (2 d - 4) (2 d - 6) );
(*c7(e1)*)
```

Bounds on two-point function

We compute bounds as explained in Section 5.1.2. We begin by computing $G_{n,z}(e_1)$ as in (5.1.22)-(5.1.24)

In[5075]:=
```
Do[
 Bound[tG7i, s] = c7i (gjz[s])^7 + (twodgjz[s])^9 * VarGamma2[s] * I110; (* G7,z(e1)*)
 Bound[tG5i, s] = c5i (gjz[s])^5 + Bound[tG7i, s]; (* G5,z(e1)*)
 Bound[tG3i, s] = c3i (gjz[s])^3 + Bound[tG5i, s]; (* G3,z(e1)*)
 Bound[tG1i, s] = gjz[s] + Bound[tG3i, s]; (* G1,z(e1)*)
 , {s, {i, o}}]
```

Then we compute $G_{n,z}(e_1 + e_2)$ and $G_{4,z}(2\, e_2)$, see (5.1.25)-(5.1.26) :

In[5076]:=
```
Do[
 Bound[tG8ik, s] = d/(d - 1) (twodgjz[o])^8 VarGamma2[s] I110; (*G8(e1+e2)*)
 Bound[tG6ik, s] = c6ik gjz[s]^6 + VarGamma2[s] I18; (*G6(e1+e2)*)
 Bound[tG4ik, s] = Bound[tG6ik, s] + (c4ik - 2 (2 d - 3) ) gjz[s]^4; (*G4^1(e1+e2)*)
 Bound[tG2ik, s] = Bound[tG4ik, s] + (c2ik - 1) gjz[s]^2; (*G2^1(e1+e2)*)
 (*Bound[tG4ii,s]=(2d+2)gjz[s]^4+Bound[tG6,s];(* Bound for sup_x G4^1(2 e1)*)*)
 , {s, {i, o}}]
```

We compute the supreme of the two-point function as given in (5.1.27)-(5.1.31):

In[5077]:=
```
Do[
 (*Bound[tG6,s]=Max[Bound[tG7i,s],Bound[tG6ik,s],(twodgjz[s])^8VarGamma2[s]I18];
 Bound for sup_x G6(x)=Max[G7(e),G6(e1+e2),sup_x G8(x)]*)
 Bound[tG6, s] = Max[c6ik gjz[s]^6, c7i gjz[s]^7] + (twodgjz[s])^8 VarGamma2[s] I18;
 (* Bound for sup_x G6(x)*)
 Bound[tG4, s] = Max[c4ik gjz[s]^4, c5i gjz[s]^5] + Bound[tG6, s];
 (* Bound for sup_x G4(x)*)
 Bound[tG2, s] = Max[c2ik gjz[s]^2, c3i gjz[s]^3] + Bound[tG4, s];
 (* Bound for sup_x G2(x)*)
 Bound[tG1, s] = Max[Bound[tG1i, s], Bound[tG2, s]]; (* Bound for sup_x G1(x)*)
 , {s, {i, o}}]
```

Closed repulsive diagrams

Then we define the bounds on the closed repulsive diagrams as desribed in Section 5.1.2 in (5.1.34)-(5.1.36). The bound does not depent on the indivitual length of the pieces m_1, m_2, ... and of the orientation of the arrows is not relevant. Thus, we use only the minimal number of steps and the number of involved two-point unfunction to label the variables

In[5078]:=
```
Do[
  Bound[ClosedRepLoop, 4, s] = twodgjz[s] Bound[tG3i, s];
  Bound[ClosedRepBubble, 4, s] =
   gjz[s]^4 (2 d c3i) + 3 gjz[s]^6 (2 d c5i) + 5 gjz[s]^8 (2 d c7i) +
    6 twodgjz[s]^10 VarGamma2[s] I110 + twodgjz[s]^10 VarGamma2[s]^2 I210;
  Bound[ClosedRepBubble, 3, s] =
   Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s];
  Bound[ClosedRepTriangle, 4, s] =
   (4 + 1 - 4) (4 + 2 - 4)/2 gjz[s]^4 (2 d c3i) + (6 + 1 - 4) (6 + 2 - 4)/2 gjz[s]^6 (2 d c5i) +
    (8 + 1 - 4) (8 + 2 - 4)/2 gjz[s]^8 (2 d c7i) +
    (10 - 4) (9 - 4)/2 twodgjz[s]^10 VarGamma2[s] I110 + 6 twodgjz[s]^10 VarGamma2[s]^2 I210 +
    twodgjz[s]^10 VarGamma2[s]^3 I310;
  Bound[ClosedRepSquare, 4, s] =
   gjz[s]^4 (2 d c3i) + 10 gjz[s]^6 (2 d c5i) + 35 gjz[s]^8 (2 d c7i) +
    84 twodgjz[s]^10 VarGamma2[s] I110 + (10 - 4) (9 - 4)/2 twodgjz[s]^10 VarGamma2[s]^2 I210 +
    6 twodgjz[s]^10 VarGamma2[s]^3 I310 + twodgjz[s]^10 VarGamma2[s]^4 I410;
  , {s, {i, o}}]
```

Open repulsive diagrams

Then we define the bound on the open repulsive diagrams as in (5.3.38):

In[5079]:=
```
Do[
 Bound[OpenRepBubble, 2, s] =
  Max[c2ik gjz[s]^2 + 3 c4ik gjz[s]^4 + 5 c6ik gjz[s]^6,
    2 c3i gjz[s]^3 + 4 c5i gjz[s]^5 + 6 c7i gjz[s]^7] + 6 twodgjz[s]^8 VarGamma2[s] I18 +
   twodgjz[s]^8 VarGamma2[s]^2 I28;
 Bound[OpenRepBubble, 3, s] =
  Max[2 c4ik gjz[s]^4 + 4 c6ik gjz[s]^6, c3i gjz[s]^3 + 3 c5i gjz[s]^5 + 5 c7i gjz[s]^7] +
   5 twodgjz[s]^8 VarGamma2[s] I18 + twodgjz[s]^8 VarGamma2[s]^2 I28;
 Bound[OpenRepTriangle, 1, s] =
  Max[3 c2ik gjz[s]^2 + 10 c4ik gjz[s]^4 + 21 c6ik gjz[s]^6,
    gjz[s] + 6 c3i gjz[s]^3 + 15 c5i gjz[s]^5 + 28 c7i gjz[s]^7] +
   ((8 - 1) (7 - 1))/2 twodgjz[s]^8 VarGamma2[s] I18 + 7 twodgjz[s]^8 VarGamma2[s]^2 I28 +
   twodgjz[s]^8 VarGamma2[s]^3 I38;
 Bound[OpenRepTriangle, 2, s] =
  Max[c2ik gjz[s]^2 + 6 c4ik gjz[s]^4 + 15 c6ik gjz[s]^6,
    3 c3i gjz[s]^3 + 10 c5i gjz[s]^5 + 21 c7i gjz[s]^7] +
   ((8 - 2) (7 - 2))/2 twodgjz[s]^8 VarGamma2[s] I18 + 6 twodgjz[s]^8 VarGamma2[s]^2 I28 +
   twodgjz[s]^8 VarGamma2[s]^3 I38;
 Bound[OpenRepTriangle, 3, s] =
  Max[3 c4ik gjz[s]^4 + 10 c6ik gjz[s]^6, c3i gjz[s]^3 + 6 c5i gjz[s]^5 + 15 c7i gjz[s]^7] +
   ((8 - 3) (7 - 3))/2 twodgjz[s]^8 VarGamma2[s] I18 + 5 twodgjz[s]^8 VarGamma2[s]^2 I28 +
   twodgjz[s]^8 VarGamma2[s]^3 I38;
 Bound[OpenRepSquare, 3, s] =
  Max[4 c4ik gjz[s]^4 + 20 c6ik gjz[s]^6, c3i gjz[s]^3 + 10 c5i gjz[s]^5 + 35 c7i gjz[s]^7] +
   56 twodgjz[s]^8 VarGamma2[s] I18 + ((8 - 3) (7 - 3))/2 twodgjz[s]^8 VarGamma2[s]^2 I28 +
   5 twodgjz[s]^8 VarGamma2[s]^3 I38 + twodgjz[s]^8 VarGamma2[s]^4 I48;
 , {s, {i, o}}]
```

In[5080]:=

Weighted Diagrams

First we define the bound on the weighted diagams in the same format as we used the in the implementation for the analysis of Section 3.3.

For $z = z_i$ we use the bound (3.6.20) :

In[5081]:=
```
Bound[WeightedClosedBubble, 4, i] = twodgjz[i]^4 BoundFFourBarInitial[1, 4, rho[i]];
Bound[WeightedClosedTriangle, 4, i] = twodgjz[i]^4 BoundFFourBarInitial[2, 4, rho[i]];

Do[
 Bound[WeightedOpenBubble, t, i] = twodgjz[i]^t BoundFThreeBarInitial[1, t, rho[i]];
 Bound[WeightedOpenTriangle, t, i] = twodgjz[i]^t BoundFThreeBarInitial[2, t, rho[i]];
 , {t, 0, 3}]
```

Then we use $f_{3,n,l}$ and $f_{4,n,l}$ that gives us direct bound for the weighted diagrams for $z \in (z_i, z_c)$:

```
In[5084]:= Bound[WeightedClosedBubble, 4, o] = twodgjz[o]^4 GammaFour[1, 4];
Bound[WeightedClosedTriangle, 4, o] = twodgjz[o]^4 GammaFour[2, 4];

Do[
 Bound[WeightedOpenBubble, t, o] = twodgjz[i]^t GammaThree[1, t];
 Bound[WeightedOpenTriangle, t, o] = twodgjz[i]^t GammaThree[2, t];
 , {t, 0, 3}]
```

Then we use the idea explained in (5.1.42)-(5.1.49) to obtain

```
In[5087]:= Do[
 Bound[WeightedClosedBubble, 2, s] =
  Bound[WeightedClosedBubble, 4, s] + 8 d gjz[s]^2 (gjz[s]^4 (2 d - 2) * 2 + Bound[tG6, s]) +
   8 d gjz[s]^2 (2 d - 2) (gjz[s]^2 + Bound[tG4ik, s]) +
   2 d gjz[s]^3 (Bound[tG3i, s] + 3 (2 d - 2) d/(d - 1) d/(d - 2) twodgjz[s]^3 VarGamma2[s] I16 +
      5 (2 d - 2) d/(d - 1) Bound[tG6, s] +
      9 (gjz[s]^2 + 3 (2 d - 2) gjz[s]^5 + Bound[tG6, s]));
 Bound[WeightedClosedTriangle, 2, s] =
  2 Bound[WeightedClosedBubble, 4, s] + Bound[WeightedClosedTriangle, 4, s] +
   8 d gjz[s]^2 (gjz[s]^4 (2 d - 2) * 2 + Bound[tG6, s]) +
   8 d gjz[s]^2 (2 d - 2) (gjz[s]^2 + Bound[tG4ik, s]) +
   4 d gjz[s]^3 (Bound[tG3i, s] + 3 (2 d - 2) d/(d - 1) d/(d - 2) twodgjz[s]^3 VarGamma2[s] I16 +
      5 (2 d - 2) d/(d - 1) Bound[tG6, s] + 9 (gjz[s]^2 + 3 (2 d - 2) gjz[s]^5 + Bound[tG6, s]));
 , {s, {i, o}}]
```

We define the elements as given in (4.3.31)-(4.3.37) and (4.3.49)-(4.3.51)

In[5088]:=
```
Do[
 Bound[Delta, I, 0, s] = 2 twodgjz[s] Bound[tG3i, s] +
   2 Bound[WeightedClosedBubble, 2, s] + Bound[WeightedClosedTriangle, 2, s];
 Bound[Delta, I, 1, s] = Bound[tG3i, s] + Bound[WeightedClosedBubble, 2, s] / twodgjz[s] +
   Bound[WeightedClosedTriangle, 2, s] / twodgjz[s];
 Bound[Delta, I, 2, s] = Bound[WeightedOpenTriangle, 0, s];
 Bound[Delta, I, 3, s] = 2 Bound[WeightedOpenBubble, 2, s] / twodgjz[s] +
   2 Bound[WeightedOpenTriangle, 3, s] / twodgjz[s];
 Bound[Delta, I, 4, s] = Bound[WeightedOpenBubble, 1, s] +
   Bound[WeightedOpenBubble, 2, s] / twodgjz[s] + 2 Bound[WeightedOpenTriangle, 2, s] / twodgjz[s];
 Bound[Delta, I, 5, s] = Bound[WeightedClosedTriangle, 2, s] / twodgjz[s]^2;
 Bound[Delta, I, 6, s] = 2 Bound[WeightedClosedBubble, 2, s] +
   Bound[WeightedClosedTriangle, 4, s];

 Bound[Delta, II, 0, s] =
  twodgjz[s] Bound[tG3i, s] + Bound[WeightedClosedBubble, 2, s] +
   Bound[WeightedClosedTriangle, 2, s];
 Bound[Delta, II, 1, s] = Bound[WeightedClosedTriangle, 2, s] / twodgjz[s];
 Bound[Delta, II, 2, s] = Bound[WeightedOpenTriangle, 1, s];
 , {s, {i, o}}]
```

Bound on the coefficients

Bound for N=0

The bounds stated in Lemma 4.3.6:

In[5089]:=
```
Do[
 Bound[Xi, normal, 0, s] = 1 + Bound[ClosedRepBubble, 4, s]; (*Bound for Ξ*)
 Bound[Xi, iota, 0, s] = rho[s] Bound[tG1, s]; (*Bound for ∑_i Ξ^i*)
 Bound[Psii, 0, s] = rho[s]; (*Bound for Ψ^k*)
 Bound[Xi, normal, 0, Delta, 0, s] = 0;

 Bound[Xi, iota, 0, Delta, 0, s] = 0;
 Bound[Xi, iota, 0, Delta, ei, s] = 2 d rho[s] Bound[tG1, s];
 , {s, {i, o}}]
```

Bound for N ≥1

Definition of Initial Pieces (P , P')

To implement the bound of N≥1 we define the boulding matricies given in (4.3.27)-(4.3.53):

In[5090]:=
```
Do[
 (* definition of first peices, inpendent of fiota*)
 Bound[P1, 0, 0, s] =
  (3 Bound[ClosedRepLoop, 4, s] + 3 Bound[ClosedRepBubble, 4, s] +
```

```
   Bound[ClosedRepTriangle, 4, s]);
Bound[P1, 0, 1, s] =
  (2 Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s] +
    Bound[ClosedRepTriangle, 4, s]);
Bound[P1, 0, 2, s] =
  (Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s] +
    Bound[ClosedRepTriangle, 4, s] + Bound[ClosedRepSquare, 4, s]);
Bound[P1, 0, -1, s] =
  (Bound[ClosedRepLoop, 4, s] + 2 Bound[ClosedRepBubble, 4, s] +
    Bound[ClosedRepTriangle, 4, s]);
Bound[P1, 0, -2, s] =
  (Bound[ClosedRepTriangle, 4, s] + Bound[ClosedRepSquare, 4, s]);;
(* definition of first peices, first step of the backbone goes to e_i*)
Do[Bound[P1, IotaStep, t, s] = Bound[P1, 0, t, s], {t, {-2, -1, 0, 1, 2}}];
Bound[P1, IotaRib, 0, s] =
  2 d (Bound[tG1i, s] Bound[P1, 0, 0, s] + Bound[ClosedRepBubble, 4, s] +
     2 Bound[tG2, s] Bound[ClosedRepLoop, 4, s] +
     Bound[tG1, s] Bound[ClosedRepBubble, 4, s] +
     Bound[tG3i, s] 1/twodgjz[s] Bound[ClosedRepBubble, 3, s] +
     (Bound[ClosedRepBubble, 4, s])/twodgjz[s] Bound[OpenRepBubble, 2, s] (*c≠x,
     d_omega=1*) + 2 Bound[OpenRepBubble, 3, s]
      (Bound[tG1i, s] + 1/twodgjz[s] Bound[ClosedRepBubble, 3, s]) (*c≠x,
     d_omega≥ 2;u≠v=x*) + Bound[OpenRepTriangle, 3, s]
      (Bound[tG1i, s] + 1/twodgjz[s] Bound[ClosedRepBubble, 3, s]) (*u≠v≠x*));
(*definition of the first piece if e_i is somewhere on the first rib*)
Bound[P1, IotaRib, 1, s] =
  2 d (Bound[tG1i, s] Bound[P1, 0, 1, s] + Bound[ClosedRepBubble, 4, s] +
     Bound[ClosedRepBubble, 4, s] + Bound[ClosedRepTriangle, 4, s] +
     1/twodgjz[s] Bound[ClosedRepBubble, 3, s] Bound[OpenRepBubble, 2, s] +
     (Bound[tG1i, s] + 1/twodgjz[s] Bound[ClosedRepBubble, 3, s])
      Bound[OpenRepTriangle, 3, s]);
Bound[P1, IotaRib, 2, s] =
  2 d (Bound[tG1i, s] Bound[P1, 0, 2, s] +
     (Bound[tG1i, s] + 1/twodgjz[s] Bound[ClosedRepBubble, 3, s])
      Bound[OpenRepSquare, 3, s]);
Bound[P1, IotaRib, -2, s] =
  2 d (Bound[tG1i, s] Bound[P1, 0, -2, s] +
```

```
      (Bound[tG1i, s] + 1/twodgjz[s] Bound[ClosedRepBubble, 3, s])
       Bound[OpenRepSquare, 3, s]);
Bound[P1, IotaRib, -1, s] =
  2 d (Bound[tG1i, s] Bound[P1, 0, -1, s] +
     Bound[tG1, s] (Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s] +
        Bound[ClosedRepTriangle, 4, s]) +
      (Bound[tG1i, s] + 1/twodgjz[s] Bound[ClosedRepBubble, 3, s])
       Bound[OpenRepTriangle, 3, s]);
, {s, {i, o}}]
```

Definition of the intermediate pieces (A $\overline{A}$)

In[5091]:=
```
Do[
 (*definition of itermediate peices, where one shared edges is counted*)
 Do[Bound[A, 0, a, s] = Bound[P1, 0, a, s], {a, {-2, -1, 0, 1, 2}}];
 Bound[A, 1, 0, s] =
  Bound[tG3i, s] +
   1/twodgjz[s] (Bound[ClosedRepLoop, 4, s] + 2 Bound[ClosedRepBubble, 4, s] +
      Bound[ClosedRepTriangle, 4, s]);
 Bound[A, 2, 0, s] = Bound[OpenRepBubble, 2, s] + Bound[OpenRepTriangle, 2, s];

 Bound[A, 1, 1, s] =
  1/twodgjz[s] (Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s] +
     Bound[ClosedRepTriangle, 4, s]);
 Bound[A, 1, 2, s] = Bound[ClosedRepTriangle, 4, s]/twodgjz[s];
 Bound[A, 2, 1, s] = Bound[OpenRepTriangle, 2, s];
 Bound[A, 2, 2, s] = Bound[OpenRepSquare, 3, s];

 Do[
  Bound[A, -t1, 0, s] = Bound[A, t1, 0, s];
  , {t1, 1, 2}];
 Clear[t1];

 (*definition of itermediate peices, where both shared edges are not counted*)
 Do[Bound[Abar, a, 0, s] = gj[s] Bound[A, a, 0, s], {a, {-2, -1, 0, 1, 2}}];
 (*(4.3.25)*)
 Do[Bound[Abar, 0, a, s] = gj[s] Bound[A, a, 0, s], {a, {-2, -1, 0, 1, 2}}];
 (*(4.3.26)*)
 Bound[Abar, 1, 1, s] =
  gj[s]
   1/(twodgjz[s])^2 (Bound[ClosedRepLoop, 4, s] + Bound[ClosedRepBubble, 4, s] +
      Bound[ClosedRepTriangle, 4, s]);
 Bound[Abar, 1, 2, s] = gj[s] Bound[OpenRepTriangle, 2, s]/twodgjz[s];
 Bound[Abar, 2, 1, s] = Bound[Abar, 1, 2, s];
 Bound[Abar, 2, 2, s] = gj[s] Bound[OpenRepTriangle, 1, s];


 (*Using Symmetrie we define the other ones*)
 Do[Do[
   Do[
    Bound[t, a, -b, s] = Bound[t, a, b, s];
    Bound[t, -a, b, s] = Bound[t, a, b, s];
    Bound[t, -a, -b, s] = Bound[t, a, b, s];
    , {a, {1, 2}}], {b, {1, 2}}], {t, {A, Abar}}];
 (*the more complex pieces*)
 , {s, {i, o}}];
```

Definition of the Delta entries

In[5092]:=
```
Do[
 Bound[Delta, start, 2, s] = Bound[Delta, I, 2, s];
 Bound[Delta, start, 1, s] = Bound[Delta, I, 1, s];
 Bound[Delta, start, 0, s] = Bound[Delta, I, 0, s];
 Bound[Delta, start, -1, s] = Bound[Delta, I, 1, s];
 Bound[Delta, start, -2, s] = Bound[Delta, I, 2, s];

 Bound[Delta, end, 2, s] = Bound[Delta, I, 4, s];
 Bound[Delta, end, 1, s] = Bound[Delta, I, 3, s];
 Bound[Delta, end, 0, s] = Bound[Delta, I, 0, s];
 Bound[Delta, end, -1, s] = Bound[Delta, I, 1, s];
 Bound[Delta, end, -2, s] = Bound[Delta, I, 2, s];

 Bound[Delta, 0, 0, s] = Bound[Delta, I, 0, s];
 Bound[Delta, 1, 0, s] = Bound[Delta, I, 3, s];
 Bound[Delta, 0, -1, s] = Bound[Delta, I, 1, s];
 Bound[Delta, -1, 0, s] = Bound[Delta, I, 1, s];
 Bound[Delta, 0, 1, s] = Bound[Delta, I, 1, s];
 Bound[Delta, -1, 1, s] = Bound[Delta, I, 5, s];
 Bound[Delta, -1, -1, s] = Bound[Delta, I, 5, s];
 Bound[Delta, 1, 1, s] = Bound[Delta, I, 6, s];
 Bound[Delta, 1, -1, s] = Bound[Delta, I, 6, s];

 Do[
  Bound[Delta, -2, t, s] = Bound[Delta, I, 2, s];
  Bound[Delta, 2, t, s] = 2 Bound[Delta, I, 2, s];
  , {t, -2, 2}];
 Bound[Delta, -1, 2, s] = Bound[Delta, I, 2, s];
 Bound[Delta, -1, -2, s] = Bound[Delta, I, 2, s];
 Bound[Delta, 1, 2, s] = 2 Bound[Delta, I, 2, s];
 Bound[Delta, 1, -2, s] = 2 Bound[Delta, I, 2, s];
 Bound[Delta, 0, 2, s] = Bound[Delta, I, 2, s];
 Bound[Delta, 0, -2, s] = Bound[Delta, I, 2, s];
 Do[
  Bound[Delta, iotaI, t, s] = Bound[Delta, II, Abs[t], s] + Bound[Delta, I, Abs[t], s] +
    Bound[tG1i, s] Bound[Delta, 0, t, s] + Bound[tG3i, s] Bound[Delta, -1, t, s] +
    Bound[ClosedRepBubble, 4, s] / twodgjz[s] Bound[Delta, -2, t, s];
  Bound[Delta, iotaII, t, s] =
   Bound[Delta, II, Abs[t], s] + Bound[Delta, I, Abs[t], s] +
    Bound[tG1i, s] Bound[Delta, 0, t, s] + 2 Bound[tG3i, s] Bound[Delta, -1, t, s] +
    2 Bound[ClosedRepBubble, 4, s] / twodgjz[s] Bound[Delta, -2, t, s] +
    2 twodgjz[s] Bound[P1, IotaRib, t, s];
  , {t, -2, 2}];
 , {s, {i, o}}]
```

Definition of the vectors and matices

We condition on the length of the backone and indentify whether the backbone is on the top or bottom of the diagram.

– the backbone in on the bottom, *d* (u, v) ≥ 2.
– the backbone in on the bottom, *d* (u, v) = 1.
u = v
– the backbone in on the top, *d* (u, v) = 1.
– the backbone in on the top, *d* (u, v) ≥ 2.

In[5093]:=
```
Do[
  VectorP1[normal, s] = Table[Bound[P1, 0, r - 3, s], {r, 1, 5}];
  VectorP1[iota, s] = Table[Bound[P1, IotaStep, r - 3, s] + Bound[P1, IotaRib, r - 3, s],
    {r, 1, 5}];

  MatrixA[s] = Table[Bound[A, r - 3, t - 3, s], {t, 1, 5}, {r, 1, 5}];
  MatrixAbar[s] = Table[Bound[Abar, r - 3, t - 3, s], {t, 1, 5}, {r, 1, 5}];
  VectorAbar[s] = Table[Bound[Abar, r - 3, 0, s], {r, 1, 5}];

  VectorDelta[start, s] = Table[Bound[Delta, start, r - 3, s], {r, 1, 5}];
  VectorDelta[end, s] = Table[Bound[Delta, end, r - 3, s], {r, 1, 5}];
  VectorDelta[iotaI, s] = Table[Bound[Delta, iotaI, r - 3, s], {r, 1, 5}];
  VectorDelta[iotaII, s] = Table[Bound[Delta, iotaII, r - 3, s], {r, 1, 5}];
  MatrixDelta[s] = Table[Bound[Delta, r - 3, t - 3, s], {t, 1, 5}, {r, 1, 5}];
  , {s, {i, o}}]
```

To compute the geometric sum over matricies we compute a representation of $P^{(1)}$ and $P^{(1)\iota}$ by eigenvalue of the matricies A:

In[5094]:=
```
Do[
  EigenA[s] = Eigensystem[Transpose[MatrixA[s]]];
  InverseProduct[normal, s] =
   Inverse[Transpose[EigenA[s][[2]]]].VectorP1[normal, s];
  InverseProduct[iota, s] = Inverse[Transpose[EigenA[s][[2]]]].VectorP1[iota, s];
  Do[
   EigenVector[normal, j, s] = EigenA[s][[2, j]] * InverseProduct[normal, s][[j]];
   EigenVector[iota, j, s] = EigenA[s][[2, j]] * InverseProduct[iota, s][[j]];
   EigenValue[j, s] = EigenA[s][[1, j]];
   , {j, 1, 5}]
  (*Print[Sum[EigenVector[j,s],{j,1,5}]-VectorP1[s]];*)
  , {s, {i, o}}]
```

Bound for k=0

Now we first implement the bound on the absolute values of the coefficients stated in Lemma 4.3.7,4.3.8 and Proposition 4.3.9

In[5095]:=
```
Do[
  Bound[Xi, normal, 0, s] = 1;
  Bound[Xi, iota, 0, s] = rho[s] Bound[tG1i, s];

  Bound[Xi, normal, 1, s] = rho[s] Bound[P1, 0, 0, s];
  Bound[Xi, iota, 1, s] = rho[s]/(2 d) Bound[P1, IotaStep, 0, s] +
    rho[s]/(2 d) Bound[P1, IotaRib, 0, s];
  factor[normal] = rho[s];
  factor[iota] = rho[s]/(2 d);
  Do[
   Bound[Xi, t, 2, s] = factor[t] VectorP1[t, s].VectorAbar[s];
   Bound[Xi, t, 3, s] =
    factor[t] VectorP1[t, s].MatrixAbar[s].VectorP1[normal, s];
   Bound[Xi, t, EvenTail, s] =
    Bound[Xi, t, 2, s] +
     Abs[
      factor[t]
       Sum[EigenVector[t, j, s].MatrixAbar[s].VectorP1[normal, s] /
          (1 - EigenValue[j, s]^2) EigenValue[j, s], {j, 1, 5}]];
   Bound[Xi, t, OddTail, s] =
    Bound[Xi, t, 3, s] +
     Abs[
      factor[t]
       Sum[EigenVector[t, j, s].MatrixAbar[s].VectorP1[normal, s] /
          (1 - EigenValue[j, s]^2) EigenValue[j, s]^2, {j, 1, 5}]];
   , {t, {normal, iota}}];
  Bound[Xi, normal, Even, s] = Bound[Xi, normal, EvenTail, s];
  (* recall here that extract the contribtion of Ξ^(0)(x)=δ_(0,x) in the analysis.*)
  Bound[Xi, iota, Even, s] = Bound[Xi, iota, 0, s] + Bound[Xi, iota, EvenTail, s];

  Do[
   Bound[Xi, t, Odd, s] = Bound[Xi, t, 1, s] + Bound[Xi, t, OddTail, s];
   Bound[Xi, t, Absolut, s] = Bound[Xi, t, Odd, s] + Bound[Xi, t, Even, s];
   , {t, {normal, iota}}];
  , {s, {i, o}}];
```

Bounds for $\hat{\Xi}^{(N)}(0)$-$\hat{\Xi}^{(N)}(k)$ for N=0,1,2,3

We now compute the bound as given in Lemma Lemma 4.3.7,4.3.8 and Proposition 4.3.9

In[5096]:=
```
Do[
 Bound[Xi, iota, 1, Delta, 0, s] = rho[s] Bound[Delta, iotaI, 0, s];
 Bound[Xi, iota, 1, Delta, ei, s] = rho[s] Bound[Delta, iotaII, 0, s];
 Bound[Xi, normal, 1, Delta, 0, s] = rho[s] Bound[Delta, I, 0, s];
 , {s, {i, o}}]
```

Bound for N=2

In[5097]:=
```
Do[
  Bound[Xi, normal, 2, Delta, 0, s] =
   2 rho[s] (VectorP1[normal, s].VectorDelta[end, s] +
      VectorDelta[start, s].VectorP1[normal, s]);
  Bound[Xi, iota, 2, Delta, 0, s] =
   2 rho[s]
    (VectorP1[iota, s].VectorDelta[end, s] +
      VectorDelta[iotaI, s].VectorP1[normal, s]);
  Bound[Xi, iota, 2, Delta, ei, s] =
   2 rho[s]
    (VectorP1[iota, s].VectorDelta[end, s] +
      VectorDelta[iotaII, s].VectorP1[normal, s]);
  , {s, {i, o}}]
```

Bound for N=3

In[5098]:=
```
Do[
  Bound[Xi, normal, 3, Delta, 0, s] =
   3 rho[s] VectorDelta[start, s].MatrixA[s].VectorP1[normal, s] +
    3 rho[s] VectorP1[normal, s].MatrixA[s].VectorDelta[end, s] +
    3 rho[s] VectorP1[normal, s].MatrixDelta[s].VectorP1[normal, s];
  Bound[Xi, iota, 3, Delta, 0, s] =
   3 rho[s] VectorDelta[iotaI, s].MatrixA[s].VectorP1[normal, s] +
    3 rho[s] VectorP1[iota, s].MatrixA[s].VectorDelta[end, s] +
    3 rho[s] VectorP1[iota, s].MatrixDelta[s].VectorP1[normal, s];
  Bound[Xi, iota, 3, Delta, ei, s] =
   3 rho[s] VectorDelta[iotaII, s].MatrixA[s].VectorP1[normal, s] +
    3 rho[s] VectorP1[iota, s].MatrixA[s].VectorDelta[end, s] +
    3 rho[s] VectorP1[iota, s].MatrixDelta[s].VectorP1[normal, s];
  , {s, {i, o}}]
```

Bounds for $\hat{\Xi}(0)$-$\hat{\Xi}(k)$ for N≥4

Bounds for $\hat{\Xi}(0)$-$\hat{\Xi}(k)$ for N>=4

We compute the sum over the bound of Proposition 4.3.9 over even N using the technique of Section 5.3.

In[5099]:=
```
Do[
  Bound[Xi, normal, EvenTail, Delta, 0, s] =
   Abs[
    rho[s] 2
     (Sum[VectorDelta[start, s].EigenVector[normal, j, s] EigenValue[j, s]^2
         (1/(1 - EigenValue[j, s]^2)^2 + 1/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
       Sum[EigenVector[normal, j, s].VectorDelta[end, s] EigenValue[j, s]^2
         (1/(1 - EigenValue[j, s]^2)^2 + 1/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
       Sum[(EigenVector[normal, j, s] EigenValue[j, s]^3) / (1 - EigenValue[j, s]^2)^2,
         {j, 1, 5}].Sum[EigenVector[normal, j, s]/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}] +
       Sum[(EigenVector[normal, j, s] EigenValue[j, s]^2) / (1 - EigenValue[j, s]^2)^2,
         {j, 1, 5}].Sum[(EigenVector[normal, j, s] EigenValue[j, s]) /
```

```
        (1 - EigenValue[j, s]^2)^2, {j, 1, 5}]]];
Bound[Xi, iota, EvenTail, Delta, 0, s] =
 Abs[
  rho[s] 2
   (Sum[VectorDelta[iotaI, s].EigenVector[normal, j, s] EigenValue[j, s]^2
       (1/(1 - EigenValue[j, s]^2)^2 + 1/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
     Sum[EigenVector[iota, j, s].VectorDelta[end, s] EigenValue[j, s]^2
       (1/(1 - EigenValue[j, s]^2)^2 + 1/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
     Sum[(EigenVector[iota, j, s] EigenValue[j, s]^3)/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
      Sum[EigenVector[normal, j, s]/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}] +
     Sum[(EigenVector[iota, j, s] EigenValue[j, s]^2)/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
      Sum[(EigenVector[normal, j, s] EigenValue[j, s]) / (1 - EigenValue[j, s]^2)^2,
       {j, 1, 5}]]];
Bound[Xi, iota, EvenTail, Delta, ei, s] =
 Abs[
  rho[s] 2
   (Sum[VectorDelta[iotaII, s].EigenVector[normal, j, s] EigenValue[j, s]^2
       (1/(1 - EigenValue[j, s]^2)^2 + 1/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
     Sum[EigenVector[iota, j, s].VectorDelta[end, s] EigenValue[j, s]^2
       (1/(1 - EigenValue[j, s]^2)^2 + 1/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
     Sum[(EigenVector[iota, j, s] EigenValue[j, s]^3)/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
      Sum[EigenVector[normal, j, s]/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}] +
     Sum[(EigenVector[iota, j, s] EigenValue[j, s]^2)/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
      Sum[(EigenVector[normal, j, s] EigenValue[j, s]) / (1 - EigenValue[j, s]^2)^2,
       {j, 1, 5}]]];
  {2, {i, 2}}];
```

Now we compute the sum over odd N

In[5100]:=
```
Do[
  Bound[Xi, normal, OddTail, Delta, 0, s] =
   Abs[
    rho[s]
     (2 Sum[VectorDelta[start, s].EigenVector[normal, j, s] EigenValue[j, s]^2
          (1/(1 - EigenValue[j, s]^2)^2 + 2/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
       2 Sum[EigenVector[normal, j, s].VectorDelta[end, s] EigenValue[j, s]^2
          (1/(1 - EigenValue[j, s]^2)^2 + 2/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
       2 Sum[(EigenVector[normal, j, s] EigenValue[j, s]^3) / (1 - EigenValue[j, s]^2)^2,
          {j, 1, 5}].Sum[EigenVector[normal, j, s]/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}] -
       Sum[(EigenVector[normal, j, s] EigenValue[j, s]^2) / (1 - EigenValue[j, s]^2)^2,
         {j, 1, 5}].Sum[EigenVector[normal, j, s] EigenValue[j, s]/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}])];
  Bound[Xi, iota, OddTail, Delta, 0, s] =
   Abs[
    rho[s]
     (2 Sum[VectorDelta[iotaI, s].EigenVector[normal, j, s] EigenValue[j, s]^2
          (1/(1 - EigenValue[j, s]^2)^2 + 2/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
       2 Sum[EigenVector[iota, j, s].VectorDelta[end, s] EigenValue[j, s]^2
          (1/(1 - EigenValue[j, s]^2)^2 + 2/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
       2 Sum[EigenVector[iota, j, s] EigenValue[j, s]^3/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
        Sum[EigenVector[normal, j, s]/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}] -
       Sum[EigenVector[iota, j, s] EigenValue[j, s]^2/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
        Sum[EigenVector[normal, j, s] EigenValue[j, s]/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}])];

  Bound[Xi, iota, OddTail, Delta, ei, s] =
   Abs[
    rho[s]
```

```
(2 Sum[VectorDelta[iotaII, s].EigenVector[normal, j, s] EigenValue[j, s]^2
      (1/(1 - EigenValue[j, s]^2)^2 + 2/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
   2 Sum[EigenVector[iota, j, s].VectorDelta[end, s] EigenValue[j, s]^2
      (1/(1 - EigenValue[j, s]^2)^2 + 2/(1 - EigenValue[j, s]^2)), {j, 1, 5}] +
   2 Sum[EigenVector[iota, j, s] EigenValue[j, s]^3/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
     Sum[EigenVector[normal, j, s]/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}] -
   Sum[EigenVector[iota, j, s] EigenValue[j, s]^2/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}].
    Sum[EigenVector[normal, j, s] EigenValue[j, s]/(1 - EigenValue[j, s]^2)^2, {j, 1, 5}])];
, {s, {i, o}}];
```

Summation of the Delta Bounds

In[5101]:=
```
Do[
 Do[
  Bound[Xi, t, Even, Delta, 0, s] =
   Bound[Xi, t, 0, Delta, 0, s] + Bound[Xi, t, 2, Delta, 0, s] +
    Bound[Xi, t, EvenTail, Delta, 0, s];
  Bound[Xi, t, Odd, Delta, 0, s] =
   Bound[Xi, t, 1, Delta, 0, s] + Bound[Xi, t, 3, Delta, 0, s] +
    Bound[Xi, t, EvenTail, Delta, 0, s];
  Bound[Xi, t, Absolut, Delta, 0, s] =
   Bound[Xi, t, Odd, Delta, 0, s] + Bound[Xi, t, Even, Delta, 0, s];
  , {t, {normal, iota}}];

 Bound[Xi, iota, Even, Delta, ei, s] =
  Bound[Xi, iota, 0, Delta, ei, s] + Bound[Xi, iota, 2, Delta, ei, s] +
   Bound[Xi, iota, EvenTail, Delta, ei, s];
 Bound[Xi, iota, Odd, Delta, ei, s] =
  Bound[Xi, iota, 1, Delta, ei, s] + Bound[Xi, iota, 3, Delta, ei, s] +
   Bound[Xi, iota, OddTail, Delta, ei, s];
 Bound[Xi, iota, Absolut, Delta, ei, s] =
  Bound[Xi, iota, Odd, Delta, ei, s] + Bound[Xi, iota, Even, Delta, ei, s];
 , {s, {i, o}}]
```

Computation of constants of Proposition 3.5.5

In this section we perform the computations of Section 3.4. to obtain the constances stated in Proposition 3.5.5:

$$\sum_x |F(x)| \le K_F = \text{Bound[KF]}$$
$$\text{Bound[KPhi, 1]} = \underline{K}_\Phi \le \hat{\Phi}(0) \le \overline{K}_\Phi = \text{Bound[KPhi, 2]}$$
$$\text{Bound[KPhiabs, 1]} = \underline{K}_{|\Phi|} \le \sum_x \left|\Phi(x)\right| \le \overline{K}_{|\Phi|} = \text{Bound[KPhiabs, 2]} \tag{3}$$
$$\sum_{x\neq 0} |\Phi_z(x)| \le K_{|\Phi|} = \text{Bound[KPhiWithoutZero]}$$

$$\sum_x F(x)[1-\cos(k\,x)] \ge K_{\text{Lower}}\left[1-\hat{D}(k)\right]$$
$$\sum_x \left|F(x)\right|[1-\cos(k\,x)] \le K_{\Delta\text{F}}\left[1-\hat{D}(k)\right] \tag{4}$$
$$\sum_x \left|\Phi_z(x)\right|[1-\cos(k\,x)] \le K_{\Delta\Phi}\left[1-\hat{D}(k)\right]$$

Bound on absolute value K_F and K_Φ

In[5102]:=
```
Do[
  alpha[s] = twodgjz[s] / (2 d);
  baralpha[s] = twodgz[s] / (2 d);
  Bound[KPsi, s] = rho[s] + (2 d - 2)/(2 d) Bound[Xi, normal, Absolut, s];
  Bound[KF, s] =
   (2 d baralpha[s]) / (1 - alpha[s] - (2 d - 2) alpha[s] Bound[Xi, iota, Absolut, s])
    Bound[KPsi, s];
  Bound[KPhiup, s] = 1 + Bound[Xi, normal, Even, s] +
    Bound[Xi, iota, Absolut, s] Bound[KF, s];
  Bound[KPhidown, s] = 1 - Bound[Xi, normal, Odd, s] -
    Bound[Xi, iota, Absolut, s] Bound[KF, s];
  Bound[KPhiabsup, s] = 1 + Bound[Xi, normal, Absolut, s] +
    Bound[Xi, iota, Absolut, s] Bound[KF, s];
  Bound[KPhiabsdown, s] = 1 - Bound[Xi, normal, Absolut, s] -
    Bound[Xi, iota, Absolut, s] Bound[KF, s];
  Bound[KPhiWithoutZero, s] =
   Bound[Xi, normal, Absolut, s] + Bound[Xi, iota, Absolut, s] Bound[KF, s];
  , {s, {i, o}}]
```

Bounds on differences

Next we implement the computation of Section 3.4.3. First the differences of $F_1$and Φ_1, lines (3.4.26), (3.4.27), (3.4.29)

```
In[5103]:= Bound[DifferencefF, Part1, Lower, i] =
   2 d baralpha[i]
   ----------------
    1 - alpha[i]^2
    (1 - Bound[tG1i, i] - Bound[Xi, normal, Odd, Delta, 0, i] - Bound[Xi, normal, Odd, i] -
      alpha[i] Bound[Xi, normal, Even, Delta, 0, i]);
  Bound[DifferencefF, Part1, Absolut, i] =
   2 d baralpha[i]
   ----------------
    1 - alpha[i]^2
    (rho[i] + (1 + alpha[i]) Bound[Xi, normal, Absolut, Delta, 0, i] +
      Bound[Xi, normal, Absolut, i]);
  Bound[DifferencefF, Part1, Lower, o] =
   Min[ 2 d baralpha[i]  ,  2 d baralpha[o] ]
       ----------------    ----------------
        1 - alpha[i]^2      1 - alpha[o]^2
    (1 - Bound[tG1i, o] - Bound[Xi, normal, Odd, Delta, 0, o] - Bound[Xi, normal, Odd, o] -
      alpha[o] Bound[Xi, normal, Even, Delta, 0, o]);
  Bound[DifferencefF, Part1, Absolut, o] =
   Max[ 2 d baralpha[i]  ,  2 d baralpha[o] ]
       ----------------    ----------------
        1 - alpha[i]^2      1 - alpha[o]^2
    (rho[i] + (1 + alpha[o]) Bound[Xi, normal, Absolut, Delta, 0, o] +
      Bound[Xi, normal, Absolut, o]);
  Do[
   Bound[KDeltaPhi, Part1, s] = Bound[Xi, normal, Absolut, Delta, 0, s] +
      baralpha[s]
     --------------
     1 - alpha[s]^2
      (2 d Bound[Xi, normal, Absolut, Delta, 0, s] Bound[Xi, iota, Absolut, s] +
        (1 + Bound[Xi, normal, Absolut, s]) Bound[Xi, iota, Absolut, Delta, ei, s] +
        2 d alpha[s] Bound[Xi, normal, Absolut, Delta, 0, s]
         Bound[Xi, iota, Absolut, s] +
        alpha[s] (1 + Bound[Xi, normal, Absolut, s])
         Bound[Xi, iota, Absolut, Delta, 0, s]);
   , {s, {i, o}}]
```

Then the differences of F_2 and Φ_2: For the bound in (3.4.30), (3.4.32) and (3.4.35)

```
In[5108]:= Do[
  Bound[DifferencefF, Part2, Lower, s] =
   -((2 d baralpha[s])^2 / (1 - alpha[s]^2))
     (Bound[Xi, normal, Odd, Delta, 0, s] Bound[Xi, iota, Odd, s] +
       Bound[Xi, normal, Even, Delta, 0, s] Bound[Xi, iota, Even, s])
    - (2 d baralpha[s]^2 / (1 - alpha[s]^2)) (rho[s] + (2 d - 2)/(2 d) Bound[Xi, normal, Even, s])
     ( Bound[Xi, iota, Even, Delta, ei, s] + 2 d Bound[Xi, iota, Even, s] +
       alpha[s]^2 Bound[Xi, iota, Even, Delta, 0, s] +
       alpha[s] (Bound[Xi, iota, Odd, Delta, ei, s] + Bound[Xi, iota, Odd, Delta, 0, s] +
          2 d Bound[Xi, iota, Odd, s])) -
    (2 d baralpha[s]^2 / (1 - alpha[s]^2)) ((2 d - 2)/(2 d) Bound[Xi, normal, Odd, s])
     ( Bound[Xi, iota, Odd, Delta, ei, s] + 2 d Bound[Xi, iota, Odd, s] +
       alpha[s]^2 Bound[Xi, iota, Odd, Delta, 0, s] +
       alpha[s] (Bound[Xi, iota, Even, Delta, ei, s] +
          Bound[Xi, iota, Even, Delta, 0, s] + 2 d Bound[Xi, iota, Even, s]));
  Bound[DifferencefF, Part2, Absolut, s] =
   ((2 d baralpha[s])^2 / (1 - alpha[s]^2)) Bound[Xi, normal, Absolut, Delta, 0, s]
     Bound[Xi, iota, Absolut, s]
    + (2 d baralpha[s]^2 / ((1 - alpha[s]^2) (1 + alpha[s]))) (rho[s] + (2 d - 2)/(2 d) Bound[Xi, normal, Absolut, s])
     ( Bound[Xi, iota, Absolut, Delta, ei, s] + 2 d Bound[Xi, iota, Absolut, s] +
       alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s]);
  Bound[KDeltaPhi, Part2, s] =
   (2 d baralpha[s]^2 / (1 - alpha[s]^2))
    (2 d Bound[Xi, normal, Absolut, Delta, 0, s] Bound[Xi, iota, Absolut, s]^2 +
     2 (1 + Bound[Xi, normal, Absolut, s])
      (Bound[Xi, iota, Absolut, s] / (1 + alpha[s]))
       (Bound[Xi, iota, Absolut, Delta, ei, s] +
         alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s]));
  , {s, {i, o}}]
```

Finally, we compute the differences of F_3 and Φ_3, lines (4.4.37) and (4.4.38)

In[5109]:=
```
Do[
 tmp = 1 / (1 - (2 d alpha[s] Bound[Xi, iota, Absolut, s]) / (1 - alpha[s]));
 Bound[DifferencefF, Part3, Absolut, s] =
  Bound[Xi, normal, Absolut, Delta, 0, s] (2 d baralpha[s]) / (1 - alpha[s])^3
     (2 d alpha[s] Bound[Xi, iota, Absolut, s])^2 tmp +
    (1 + Bound[Xi, normal, Absolut, s]) (baralpha[s] (2 d alpha[s])^2) / ((1 - alpha[s])^2 (1 - alpha[s]^2))
     Bound[Xi, iota, Absolut, s] tmp^2
     (Bound[Xi, iota, Absolut, Delta, ei, s] +
       alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s]) +
    (1 + Bound[Xi, normal, Absolut, s]) (baralpha[s] (2 d alpha[s])^2) / ((1 - alpha[s])^2 (1 - alpha[s]^2))
     Bound[Xi, iota, Absolut, s] tmp
     (Bound[Xi, iota, Absolut, Delta, ei, s] +
       alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s] +
       2 d Bound[Xi, iota, Absolut, s]);

 Bound[DifferencefF, Part3, Lower, s] = -Bound[DifferencefF, Part3, Absolut, s];
 Bound[KDeltaPhi, Part3, s] =
  Bound[Xi, normal, Absolut, Delta, 0, s] (2 d baralpha[s] Bound[Xi, iota, Absolut, s]) / (1 - alpha[s])^3
     (2 d alpha[s] Bound[Xi, iota, Absolut, s])^2 tmp +
    (1 + Bound[Xi, normal, Absolut, s])
     (baralpha[s] (2 d alpha[s] Bound[Xi, iota, Absolut, s])^2) /
      ((1 - alpha[s])^2 (1 - alpha[s]^2)) (tmp^2 + tmp)
     1 / (1 + alpha[s]) (Bound[Xi, iota, Absolut, Delta, ei, s] +
        alpha[s] Bound[Xi, iota, Absolut, Delta, 0, s]);

 Bound[KDeltaFLower, s] =
  1 / (Bound[DifferencefF, Part1, Lower, s] + Bound[DifferencefF, Part2, Lower, s] +
      Bound[DifferencefF, Part3, Lower, s]);
 Bound[KDeltaF, s] = Bound[DifferencefF, Part1, Absolut, s] +
   Bound[DifferencefF, Part2, Absolut, s] + Bound[DifferencefF, Part3, Absolut, s];
 Bound[KDeltaPhi, s] = Bound[KDeltaPhi, Part1, s] + Bound[KDeltaPhi, Part2, s] +
   Bound[KDeltaPhi, Part3, s];
 Clear[tmp];
 , {s, {i, o}}]
```

Computation of additial bounds of Assumption 3.5.3

Now we compute bounds on α_F, α_Φ, $R_F$and R_Φ as given in Section 3.5.3 and 4.3.9. We follow the structure of Section 4.3.9. We know that $z \geq z_I = e^{-1}/(2d-1)$ and by simple combinatorics that

$$g_z^i \geq 1 + (2d-1)\, z_I(1 + (2d-1)\, z_I) + (2d-1)(2d-2)\, \frac{z_I^2}{2}$$

```
In[5110]:= z[i] = 1/((2 d - 1) Exp[1]);
gjzLower = 1 + (2 d - 1) z[i] (1 + (2 d - 1) z[i]) + ((2 d - 1) (2 d - 2))/2 z[i]^2;
gjjzLower = 1 + (2 d - 2) z[i] (1 + (2 d - 1) z[i]) + (2 d - 2)^2/2 z[i]^2;
gzLower = 1 + 2 d * z[i] + 2 × 2 d (2 d - 1) * z[i] ^2 +
   2 d * (6 d - 4 + 6 d - 6 + (2 d - 2) / 2 (6 d - 8) + (2 d - 2) (6 d - 8)) z[i] ^3 +
   (2 d (8 d - 6 + 8 d - 8) +
      2 d (2 d - 2) (1 / 2 + 8 d - 9 + 6 d - 6 + 4 d - 4 + 4 d - 4 + 4 d - 2 + 8 d - 12 + 6 d - 8) +
      2 d (2 d - 2) (2 d - 4) (2 d - 6) / 4! + 2 d (2 d - 2) (2 d - 4) / 2 * (2 d - 6)) z[i] ^4;
z[o] = Gamma1/((2 d - 1) * gzLower);(* Upper bound on z and thereby also on z_c *)
alphaLower = z[i] gjzLower;
```

To rewrite Φ as in (3.5.31)-(3.5.33) we extract from $\Xi^{(0)}$ and $\Xi^{(1)}$ the nearest neighbor contribtion. For the implementation we split $\Xi^{(0)}$, $\Xi^{(1)}$ and $\Xi^{(0),i}$ as follows

```
In[5116]:= Bound[Xi, normalalphaPhi, 0, o] = 0;
Bound[Xi, normalRPhi, 0, o] = 0;

Bound[Xi, normalalphaPhi, 1, o] =
  rho[o] (2 Bound[ClosedRepLoop, 4, o] + Bound[ClosedRepBubble, 4, o]);
Bound[Xi, normalRPhi, 1, o] =
  rho[o] (Bound[ClosedRepLoop, 4, o] + 2 Bound[ClosedRepBubble, 4, o] +
     Bound[ClosedRepTriangle, 4, o]);


Bound[Xi, iotaalphaPhi, 0, o] = rho[o] Bound[tG1, o];
Bound[Xi, iotaRPhi, 0, o] = 0;
Bound[Xi, iotaRPhi, 0, DeltaPhi, 0, o] = 0;
Bound[Xi, iotaRPhi, 0, DeltaPhi, ei, o] = 0;

In[5124]:= Bound[Xi, normalalphaPhi, 1, o]

Out[5124]= 0.00316336
```

We use these quantities to define the bounds

```
In[5125]:= ap = Max[Bound[Xi, normalalphaPhi, 0, o],
    Bound[Xi, normalalphaPhi, 1, o] + twodgz[o]/(1 - gjz[o]^2) Bound[Xi, iotaalphaPhi, 0, o]];
  Bound[Phi2Phi3, Absolut, 0, o] =
    (twodgz[o]/(1 - gjz[o]))^2 (d - 1)/d Bound[KPsi, o] Bound[Xi, iota, Absolut, o]^2
    1 / (1 - alpha[o] - (2 d - 2) alpha[o] Bound[Xi, iota, Absolut, o]);
  (*compare with (3.4.11)*)

  bRp = Bound[Xi, normalRPhi, 0, o] + Bound[Xi, normalRPhi, 1, o] +
    Sum[Bound[Xi, normal, t, o], {t, {2, 3, EvenTail, OddTail}}] +
    twodgjz[o]/(1 - gjz[o]^2)
     (Bound[Xi, iotaRPhi, 0, o] + Sum[Bound[Xi, normal, t, o],
        {t, {1, 2, 3, EvenTail, OddTail}}]) +
    twodgz[o]/(1 - gjz[o]^2) Bound[Xi, normal, Absolut, o] Bound[Xi, iota, Absolut, o] +
    Bound[Phi2Phi3, Absolut, 0, o];
```

see (3.5.32)-(3.5.33).

To compute $\Phi(0) - \Phi(k)$ we compute the remainder term for the difference $\Phi_1(0) - \Phi_1(k)$:

```
In[5128]:= Bound[Xi, normalRPhi, 0, Delta, 0, o] = 0;
  Bound[Xi, normalRPhi, 1, Delta, 0, o] =
   2 Bound[WeightedClosedBubble, 2, o] + Bound[WeightedClosedTriangle, 2, o];

  Bound[Xi, iotaRPhi, Absolut, o] =
   Bound[Xi, iota, Absolut, o] - Bound[Xi, iota, 0, o] + Bound[Xi, iotaRPhi, 0, o];
  Bound[Xi, iotaRPhi, Absolut, Delta, 0, o] =
   Bound[Xi, iota, Absolut, Delta, 0, o] - Bound[Xi, iota, 0, Delta, 0, o] +
    Bound[Xi, iotaRPhi, 0, DeltaPhi, 0, o];
  Bound[Xi, iotaRPhi, Absolut, Delta, ei, o] =
   Bound[Xi, iota, Absolut, Delta, ei, o] - Bound[Xi, iota, 0, Delta, ei, o] +
    Bound[Xi, iotaRPhi, 0, DeltaPhi, ei, o];
```

Then, we use these differences and add the already computed differences $\Phi_2(0) - \Phi_2(k)$ and $\Phi_3(0) - \Phi_3(k)$:

```
In[5133]:= bRpDelta = Bound[Xi, normalRPhi, 0, Delta, 0, o] +
     Bound[Xi, normalRPhi, 1, Delta, 0, o] +
     Sum[Bound[Xi, normal, t, Delta, 0, o], {t, {2, 3, EvenTail, OddTail}}] +
      baralpha[o]
     ─────────────
     1 - alpha[o]^2
      (2 d Bound[Xi, normal, Absolut, Delta, 0, o] Bound[Xi, iotaRPhi, Absolut, o] +
        (1 + Bound[Xi, normal, Absolut, o]) Bound[Xi, iotaRPhi, Absolut, Delta, ei, o] +
       2 d alpha[o] Bound[Xi, normal, Absolut, Delta, 0, o]
         Bound[Xi, iotaRPhi, Absolut, o] +
       alpha[o] (1 + Bound[Xi, normal, Absolut, o])
         Bound[Xi, iotaRPhi, Absolut, Delta, 0, o]) +
      baralpha[o]
     ─────────────
     1 - alpha[o]^2
      (2 d alpha[o] Bound[Xi, normal, Absolut, Delta, 0, o]
         Bound[Xi, iotaalphaPhi, 0, o]
         ─────────────────────────────  +
                    2 d
       alpha[o] Bound[Xi, normal, Absolut, o] Bound[Xi, iotaalphaPhi, 0, o]) +
     Bound[KDeltaPhi, Part2, o] + Bound[KDeltaPhi, Part3, o];
```

The first term corresponds are contributions to Ξ_z that have not been extracted. The second term bound $\sum_i \Psi^{(0),i}(0)\,(\hat{\Xi}^i(0) - \Xi^i(e_i)\big)$. The third term bounds $\sum_i \left(\hat{\Psi}^{(0),i}(0) - \Psi^{(0),i}(0)\right)\Xi^i(e_i)\big)$. In last term bound all remainder term the contribution of $\Phi_2$and Φ_3.

For the rewrite of $1 - F(k)$ we require the following quantities:

```
In[5134]:= Psi1eiekLower = 6 z[i]^4 gjjzLower^4 + 20 (2 d - 2) z[i]^6 gjjzLower^6;(*Ψ^{1,k}(e1+e2)*)
           Psi0e1Lower = 0;(*Ψ^{0,k}(e1)*)
           Psi0e2Lower = 0;(*Ψ^{0,k}(e2)*)

           Psi0eiek = 0;(*Ψ^{0,k}(ei+ek)*)
                     Bound[ClosedRepBubble, 3, o]
           Psi1e2 = ──────────────────────────────── ;
                                2 d
           Psi1e1 = Psi1e2;
```

We use as bound on the absolute value of α_F(3.5.39) the following

```
                     gjz[o]                      (2 d - 2)
In[5140]:= af = 2 d ─────────── + twodgz[o] ─────────── (Psi0eiek - Psi1eiekLower) +
                   1 - gjz[o]^2               1 - gjz[o]^2
                           gjz[o]
              twodgz[o] ─────────── ((2 d - 2) (Psi1e2 - Psi0e2Lower) + (Psi1e1 - Psi0e1Lower));
                         1 - gjz[o]^2
```

For the computation of $F(0) - F(k)$ we use the following values

In[5141]:=
```
Bound[Xi, normalRf, 0, Delta, 0, o] = 0;
Bound[Xi, normalRf, 1, Delta, 0, o] =
  2 Bound[WeightedClosedBubble, 2, o] + Bound[WeightedClosedTriangle, 2, o];
Bound[Xi, normalRf, 0, o] = 0;
Bound[Xi, normalRf, 1, o] =
  (Bound[ClosedRepLoop, 4, o] + 2 Bound[ClosedRepBubble, 4, o] +
    Bound[ClosedRepTriangle, 4, o]);
Bound[Xi, normalRf, Absolut, o] =
  Bound[Xi, normal, Absolut, o] + Bound[Xi, normalRf, 0, o];
Bound[Xi, normalRf, Absolut, Delta, 0, o] =
  Bound[Xi, normal, Absolut, Delta, 0, o] +
   Sum[Bound[Xi, normalRf, t, Delta, 0, o] - Bound[Xi, normal, t, Delta, 0, o],
    {t, 0, 1}];
```

to create the bound s

In[5147]:=

$$\texttt{bRf} = \frac{\texttt{twodgz[o]}}{1-\texttt{gjz[o]}}\,\frac{2d-2}{2d}\,\texttt{Bound[Xi, normalRf, Absolut, o]} + \frac{\texttt{twodgz[o]}}{1-\texttt{gjz[o]}^2}\,\texttt{Bound[KPsi, o]}\,\frac{\frac{2d-2}{2d}\,\frac{\texttt{twodgz[o]}}{1-\texttt{gjz[o]}}\,\texttt{Bound[Xi, iota, Absolut, o]}}{1-\frac{\texttt{twodgz[o]}}{1-\texttt{gjz[o]}}\,\texttt{Bound[Xi, iota, Absolut, o]}};$$

$$\texttt{bRfDelta} = \frac{2\,d\,\texttt{baralpha[o]}}{1-\texttt{alpha[o]}^2}\,(\texttt{Bound[Xi, normalRf, Absolut, Delta, 0, o]} + \texttt{Bound[Xi, normalRf, Absolut, o]} + \texttt{alpha[o] Bound[Xi, normalRf, Absolut, Delta, 0, o]}) + \texttt{Bound[DifferencefF, Part2, Absolut, o]} + \texttt{Bound[DifferencefF, Part3, Absolut, o]};$$

Check of the sufficient condition

Now we can compute whether Q(γ,Γ,z) is satisfied, see Definition 3.5.6.

In[5149]:= Do[

$$\texttt{NoBLEBoundF1[s]} = \frac{1+\frac{2d-2}{2d-1}\,\texttt{Gamma1 Bound[Xi, iota, Even, s]}}{\texttt{rho[s]} - \frac{2d-2}{2d}\,\texttt{Bound[Xi, normal, Odd, s]}};$$

$$\texttt{NoBLEBoundF2[s]} = \frac{2d-2}{2d-1}\,\texttt{Bound[KPhiabsup, s] Bound[KDeltaFLower, s]};$$

, {s, {i, o}}]

We finally check

In[5150]:=
```
Do[
 Succes[f1, s] = NoBLEBoundF1[s] < Gamma1;
 Succes[f2, s] = NoBLEBoundF2[s] < Gamma2;
 Succes[s] = Succes[f1, s] && Succes[f2, s];
 , {s, {i, o}}]
```

Further, we need the constants for the improvement of $\bar{f}_{3,n,l}$ and $\bar{f}_{4,n,l}$:

```
In[5151]:= BoundFFour[n_, l_] := BoundFFourBar[n, l, (2 d - 2)/(2 d - 1) Gamma2, twodgz[s], 1, af,
              ap, bRf, bRp, bRfDelta, bRpDelta, Bound[KDeltaFLower, o]];
            BoundFThree[n_, l_] := BoundFThreeBar[n, l, (2 d - 2)/(2 d - 1) Gamma2, twodgz[s], 1, af,
              ap, bRf, bRp, bRfDelta, bRpDelta, Bound[KDeltaFLower, o]];

In[5153]:= Do[
             SuccesFThree[t + 1] = BoundFThree[1, t] < GammaThree[1, t];
             SuccesFThree[t + 5] = BoundFThree[2, t] < GammaThree[2, t];
             , {t, 0, 3}]
            Succes[f3bar] = SuccesFThree[1] && SuccesFThree[2] && SuccesFThree[3] &&
              SuccesFThree[4] && SuccesFThree[5] && SuccesFThree[6] && SuccesFThree[7] &&
              SuccesFThree[8];

            SuccesFFour[1] = BoundFFour[1, 4] < GammaFour[1, 4];
            SuccesFFour[2] = BoundFFour[2, 4] < GammaFour[2, 4];
            Succes[f4bar] = SuccesFFour[1] && SuccesFFour[2];
            Succes[overall] = Succes[i] && Succes[o] && Succes[f3bar] && Succes[f4bar];
```

Result

The overall result

The statement that the bootstrap was succesful is

```
In[5159]:= Succes[overall]

Out[5159]= True
```

If this succedes than the analysis of Section 3.5 can be used to proved mean-field behavoior for LT.

Assuming the bootstrap was succesful we prove the following bounds: The infrared bound in (3.5.28) and (3.5.29) hold with

```
In[5160]:= (2 d - 2)/(2 d - 1) Gamma2 (* ≥ G_z(k) [1-D̂(k)] *)
           Max[Bound[KDeltaFLower, o], 1]
           (* Nominator in (4.5.29) *)

Out[5160]= 1.11398

Out[5161]= 1.12417
```

Further, we have proven that $g_{z_c} z_c$ is smaller than

```
In[5162]:= 1/(2 d - 1) Gamma1

Out[5162]= 0.0270419
```

and that g_{z_c} smaller than

```
In[5163]:= Gamma1 * Exp[1]

Out[5163]= 2.8668
```

The improvement of bounds

In[5164]:=
```
bubbles = {Graphics[{Green, Disk[{0, 0}, 0.8]}, ImageSize -> 30],
   Graphics[{Red, Disk[{0, 0}, 0.8]}, ImageSize -> 40]};
tableClassicCheck = {{Bounds, Init - f_1, Init - f_2, f_1, f_2, f̄_{4,1,4}, f̄_{4,2,4}},
   {Gamma, Gamma1, Gamma2, Gamma1, Gamma2 , GammaFour[1, 4], GammaFour[2, 4] },
   {Bounds, NoBLEBoundF1[i], NoBLEBoundF2[i], NoBLEBoundF1[o], NoBLEBoundF2[o],
    BoundFFour[1, 4], BoundFFour[2, 4]}, { check,
    If[Succes[f1, i], bubbles[[1]], bubbles[[2]]],
    If[Succes[f2, i], bubbles[[1]], bubbles[[2]]],
    If[Succes[f1, o], bubbles[[1]], bubbles[[2]]],
    If[Succes[f2, o], bubbles[[1]], bubbles[[2]]],
    If[SuccesFFour[1], bubbles[[1]], bubbles[[2]]],
    If[SuccesFFour[2], bubbles[[1]], bubbles[[2]]]}};
tableClassicCheckFthree =
  {{Bounds, "(1,0)", "(1,1)", "(1,2)", "(1,3)", "(2,0)", "(2,1)", "(2,2)", "(2,3)"},
   {Gamma, GammaThree[1, 0], GammaThree[1, 1], GammaThree[1, 2],
    GammaThree[1, 3] , GammaThree[2, 0], GammaThree[2, 1], GammaThree[2, 2],
    GammaThree[2, 3] }, {Bounds, BoundFThree[1, 0], BoundFThree[1, 1],
    BoundFThree[1, 2], BoundFThree[1, 3], BoundFThree[2, 0], BoundFThree[2, 1],
    BoundFThree[2, 2], BoundFThree[2, 3]}, { check,
    If[SuccesFThree[1], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[2], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[3], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[4], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[5], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[6], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[7], bubbles[[1]], bubbles[[2]]],
    If[SuccesFThree[8], bubbles[[1]], bubbles[[2]]]}};
Labeled[Grid[tableClassicCheck, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {Automatic, Automatic, {{3, 3}, {2, 7}} → GrayLevel[0.7]}],
  Style["Result for f_1 and f_2  in  Dimension " Text[d], Bold], Top] // Text
Labeled[Grid[tableClassicCheckFthree, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True, 5 → True}},
   ItemStyle → {1 → Bold, 1 → Bold},
   Background → {Automatic, Automatic, {{3, 3}, {2, 9}} → GrayLevel[0.9]}],
  Style["Result for f_3 in  Dimension " Text[d], Bold], Top] // Text
```

Out[5167]=

Result for f_1 and f_2 in Dimension 20

Bounds	**Init − f_1**	**Init − f_2**	**f_1**	**f_2**	**$\bar{f}_{4,1,4}$**	**$\bar{f}_{4,2,4}$**
Gamma	1.05464	1.1433	1.05464	1.1433	0.003234	0.0067444
Bounds	1.05012	1.0845	1.05464	1.14329	0.0032333	0.00674417
check						

Out[5168]=

Result for f_3 in Dimension 20

Bounds	**(1,0)**	**(1,1)**	**(1,2)**	**(1,3)**	**(2,0)**	**(2,1)**	**(2,2)**	**(2,3)**
Gamma	0.1367	0.026228	0.010843	0.003234	0.20456	0.04752	0.01965	0.00675
Bounds	0.136655	0.0262258	0.0108418	0.0032333	0.204548	0.047519	0.0196496	0.00674417
check								

In the following we implement a semi-automate proceedure to find appropiate value for the constants Γ_i. Initially we guess a good value for the constant and make a first computation. Then we deactivate there initial definition in the top of the document and use the code below.

We recompile the document a number of times and hope that the algorithm below converges to a fixed point.

In[5169]:=
```
{d, Gamma1, Gamma2, GammaThree[1, 0], GammaThree[1, 1], GammaThree[1, 2],
 GammaThree[1, 3], GammaThree[2, 0], GammaThree[2, 1], GammaThree[2, 2],
 GammaThree[2, 3], GammaFour[1, 4], GammaFour[2, 4]}
Gamma1 = NoBLEBoundF1[o] + 0.000001;
Gamma2 = NoBLEBoundF2[o] + 0.000001;
Do[
 GammaThree[1, t] = BoundFThree[1, t] + 0.000001;
 GammaThree[2, t] = BoundFThree[2, t] + 0.000001;
 , {t, 0, 3}]
GammaFour[1, 4] = BoundFFour[1, 4] + 0.000001;
GammaFour[2, 4] = BoundFFour[2, 4] + 0.000001;
{d, Gamma1, Gamma2, GammaThree[1, 0], GammaThree[1, 1], GammaThree[1, 2],
 GammaThree[1, 3], GammaThree[2, 0], GammaThree[2, 1], GammaThree[2, 2],
 GammaThree[2, 3], GammaFour[1, 4], GammaFour[2, 4]}
```

Out[5169]=
```
{20, 1.05464, 1.1433, 0.1367, 0.026228, 0.010843,
 0.003234, 0.20456, 0.04752, 0.01965, 0.00675, 0.003234, 0.0067444}
```

Out[5175]=
```
{20, 1.05464, 1.14329, 0.136655, 0.0262266, 0.0108428, 0.00323428,
 0.204547, 0.0475194, 0.0196504, 0.0067451, 0.00323428, 0.0067451}
```

Print out of the computed bounds in the coefficients

In[5176]:=
```
Do[
 MethodeFourTable[s] = {{Quantity, Ξ^Zero, Ξ^One, Ξ^Two, Ξ^Three, Ξ^EvenTail, Ξ^OddTail},
   {Text[Bound for Ξ̂], Bound[Xi, normal, 0, s], Bound[Xi, normal, 1, s] ,
    Bound[Xi, normal, 2, s], Bound[Xi, normal, 3, s], Bound[Xi, normal, EvenTail, s],
    Bound[Xi, normal, OddTail, s]},
   {Text[Bound for Ξ̂^ι], Bound[Xi, iota, 0, s], Bound[Xi, iota, 1, s] ,
    Bound[Xi, iota, 2, s], Bound[Xi, iota, 3, s], Bound[Xi, iota, EvenTail, s],
    Bound[Xi, iota, OddTail, s]},
   {Text[Ξ̂ (1 - cos (kx))], Bound[Xi, normal, 0, Delta, 0, s],
    Bound[Xi, normal, 1, Delta, 0, s], Bound[Xi, normal, 2, Delta, 0, s],
    Bound[Xi, normal, 3, Delta, 0, s], Bound[Xi, normal, EvenTail, Delta, 0, s],
    Bound[Xi, normal, OddTail, Delta, 0, s]},
   {Text[ Ξ^ι (1 - cos (kx))], Bound[Xi, iota, 0, Delta, 0, s],
    Bound[Xi, iota, 1, Delta, 0, s], Bound[Xi, iota, 2, Delta, 0, s],
    Bound[Xi, iota, 3, Delta, 0, s], Bound[Xi, iota, EvenTail, Delta, 0, s],
    Bound[Xi, iota, OddTail, Delta, 0, s]},
   {Text[Ξ^ι (1 - cos (k (x - e_ι)))] , Bound[Xi, iota, 0, Delta, ei, s],
    Bound[Xi, iota, 1, Delta, ei, s] , Bound[Xi, iota, 2, Delta, ei, s],
    Bound[Xi, iota, 3, Delta, ei, s], Bound[Xi, iota, EvenTail, Delta, ei, s],
    Bound[Xi, iota, OddTail, Delta, ei, s]}};
 , {s, {i, o}}]
MethodeFourTablePart1 = {{Quantity, KF, KPhi, DELTAFLower, DELTAFAbsolut, DELTAPhi},
  {Bound for , Bound[KF, o], Bound[KPhiup, o], Bound[KDeltaFLower, o],
   Bound[KDeltaF, o], Bound[KDeltaPhi, o]}};
MethodeFourTablePart2 =
  {{Quantity, DELTAFLower, 2, 3, DELTAFAbsolut, 2, 3, DELTAPhi, 2, 3},
   {Bound for , Bound[DifferencefF, Part1, Lower, o],
    Bound[DifferencefF, Part2, Lower, o], Bound[DifferencefF, Part3, Lower, o],
    Bound[DifferencefF, Part1, Absolut, o], Bound[DifferencefF, Part2, Absolut, o],
    Bound[DifferencefF, Part3, Absolut, o], Bound[KDeltaPhi, Part1, o],
    Bound[KDeltaPhi, Part2, o], Bound[KDeltaPhi, Part3, o]}};
```

```
MethodeFourTablePart3 = {{Quantity, αΦ, RΦ, ΔRΦ, αF, RF, ΔRF},
   {Bound for , ap, bRp, bRpDelta, af, bRf, bRfDelta}};


Labeled[Grid[MethodeFourTable[i], Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Bound on coefficients at zI in Dimension " Text[d], Bold], Top] // Text
Labeled[Grid[MethodeFourTable[o], Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Bound on coefficients in Dimension " Text[d], Bold], Top] // Text

Labeled[Grid[MethodeFourTablePart1, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Bound on the constants of Proposition 3.5.5 in Dimension " Text[d], Bold],
  Top] // Text
Labeled[Grid[MethodeFourTablePart2, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Bound on the constants of Proposition 3.5.5 in Dimension " Text[d], Bold],
  Top] // Text
Labeled[Grid[MethodeFourTablePart3, Alignment → {Center}, Frame → True,
   Dividers → {{2 → True, -1 → True}, {2 → True}}, ItemStyle → {1 → Bold, 1 → Bold},
   Background → {{None}, {GrayLevel[0.9]}, {None}}],
  Style["Element of the rewrite of the two-point function " Text[d], Bold], Top] //
 Text
```

Bound on coefficients at z_I in Dimension 20

Out[5180]=

Quantity	Ξ^{Zero}	Ξ^{One}	Ξ^{Two}	Ξ^{Three}	$\Xi^{EvenTail}$	$\Xi^{OddTail}$
Bound for $\hat{\Xi}$	1	0.00660959	0.000415675	0.0000184303	0.000416126	0.0000184406
Bound for $\hat{\Xi}^{\iota}$	0.0259444	0.00147198	0.0000962558	2.87827×10^{-6}	0.0000963581	2.88068×10^{-6}
$(1-\cos kx)\hat{\Xi}$	0	0.0166604	0.00233819	0.000171106	2.61049×10^{-6}	3.90699×10^{-6}
$(1-\cos kx)\,\Xi^{\iota}$	0	0.0289835	0.00926059	0.00116371	0.0000128548	0.0000192016
$\Xi^{\iota}\,(1-\cos k\,(x-e_{\iota}))$	1.03777	0.13625	0.0138544	0.00132842	0.0000170328	0.0000254681

Bound on coefficients in Dimension 20

Out[5181]=

Quantity	Ξ^{Zero}	Ξ^{One}	Ξ^{Two}	Ξ^{Three}	$\Xi^{EvenTail}$	$\Xi^{OddTail}$
Bound for $\hat{\Xi}$	1	0.00897727	0.000877499	0.0000562889	0.000879581	0.0000563623
Bound for $\hat{\Xi}^{\iota}$	0.0274557	0.00204096	0.000196775	8.97369×10^{-6}	0.000197215	8.98966×10^{-6}
$(1-\cos kx)\hat{\Xi}$	0	0.0401686	0.0150219	0.00182523	0.0000407369	0.0000610232
$(1-\cos kx)\,\Xi^{\iota}$	0	0.0725346	0.0520843	0.0119694	0.000180838	0.000270593
$\Xi^{\iota}\,(1-\cos k\,(x-e_{\iota}))$	1.09823	0.230001	0.0621877	0.0125215	0.000202395	0.000302923

Bound on the constants of Proposition 3.5.5 in Dimension 20

Out[5182]=

Quantity	KF	KPhi	DELTAFLower	DELTAFAbsolut	DELTAPhi
Bound for	1.13987	1.03474	1.12417	1.21913	0.0999931

Bound on the constants of Proposition 3.5.5 in Dimension 20

Out[5183]=

Quantity	DELTA⋱ FLow⋱ er	2	3	DELTA⋱ FAbs⋱ olut	2	3	DELTA⋱ Phi	2	3
Bound for	0.960155	−0.0663⋱ 334	−0.0042⋱ 743⋱ 2	1.13941	0.07544⋱ 65	0.00427⋱ 432	0.09744	0.00246⋱ 509	0.00008⋱ 80195

Element of the rewrite of the two−point function 20

Out[5184]=

Quantity	α_Φ	R_Φ	ΔR_Φ	α_F	R_F	ΔR_F
Bound for	0.0328833	0.0208356	0.0670442	1.08252	0.0453464	0.15258